4

INTRODUCTION TO LINEAR PROGRAMMING

(FORMULATION, GRAPHICAL, AND ANALYTIC METHODS)

4.1 INTRODUCTION

In 1947, George Dantzig and his Associates, while working in the U.S. department of Air Force, observed that a large number of military programming and planning problems could be formulated as maximizing/minimizing a linear form of profit/cost function whose variables were restricted to values satisfying a system of linear constraints (a set of linear equations/or inequalities). A linear form is meant a mathematical expression of the type $a_1x_1 + a_2x_2 + \ldots + a_nx_n$, where $a_1, a_2, \ldots, a_n$ are constants, and $x_1, x_2, \ldots, x_n$ are variables. The term '*Programming*' refers to the process of determining a particular programme or plan of action. So Linear Programming (L.P.) is one of the most important optimization (maximization/minimization) techniques developed in the field of Operations Research (O.R.).

The methods applied for solving a linear programming problem are basically simple problems, a solution can be obtained by a set of simultaneous equations. However, a ***unique*** solution for a set of simultaneous equations in n-variables $(x_1, x_2, \ldots, x_n)$, at least one of them is non-zero, can be obtained if there are exactly n relations. When the number of relations is greater than or less than n, a unique solution does not exist, but a number of trial solutions can be found. In various practical situations, the problems are seen in which the number of relations is not equal to the number of variables and many of the relations are in the form of inequalities ($\leq$ or $\geq$) to maximize (or minimize) a linear function of the variables subject to such conditions. Such problems are known as *Linear Programming Problems* (LPP).

In this chapter, properties of LP problems are discussed and at present the graphical method of solving a LPP is applicable where two (or at most three) variables are involved. The most widely used method for solving LP problems of any number of variables is called the *simplex method* developed by G. Dantzig in 1947 and made generally available in 1951.

Definition. *The general LPP calls for optimizing (maximizing/minimizing) a linear function of variables called the 'OBJECTIVE FUNCTION' subject to a set of linear equations and / or inequalities called the 'CONSTRAINTS' or 'RESTRICTIONS'.*

4.2 FORMULATION OF LP PROBLEMS

Now it becomes necessary to present a few interesting examples to explain the real-life situations where LP problems may arise. The outlines of formulation of the LP problems are explained with the help of these examples.

Example 1. (Production Allocation Problem) *A firm manufactures two type of products A and B and sells them at a profit of Rs. 2 on type A and Rs. 3 on type B. Each product is processed on two machines G and H. Type A requires one minute of processing time on G and two minutes on H ; type B requires one minute on G and one minute on H. The machine G is available for not more than 6 hour 40 minutes while machine H is available for 10 hours during any working day.*

Formulate the problem as a linear programming problem. **[Kanpur 96]**

Formulation. Let x_1 be the number of products of type A and x_2 the number of products of type B.

After carefully understanding the problem the given information can be systematically arranged in the form of the following table.

Table 4.1

Machine	Time of Products (minutes)		Available Time (minutes)
	Type *A* (x_1 units)	Type *B* (x_2 units)	
G	1	1	400
H	2	1	600
Profit per unit	Rs. 2	Rs. 3	

Since the profit on type *A* is Rs. 2 per product, $2x_1$ will be the profit on selling x_1 units of type *A*. Similarly, $3x_2$ will be the profit on selling x_2 units of type *B*. Therefore, total profit on selling x_1 units of *A* and x_2 units of *B* is given by

$$P = 2x_1 + 3x_2 \quad \textit{(objective function)}$$

Since machine *G* takes 1 minute time on type *A* and 1 minute time on type *B*, the total number of minutes required on machine *G* is given by : $x_1 + x_2$.

Similarly, the total number of minutes required on machine *H* is given by $2x_1 + x_2$.

But, machine *G* is not available for more than 6 hour 40 minutes (= 400 minutes). Therefore,

$$x_1 + x_2 \le 400 \quad \textit{(first constraint)}$$

Also, the machine *H* is available for 10 hours only, therefore,

$$2x_1 + x_2 \le 600 \quad \textit{(second constraint)*}$$

Since it is not possible to produce negative quantities,

$$x_1 \ge 0 \text{ and } x_2 \ge 0 \quad \textit{(non-negativity restrictions)}$$

Hence the allocation problem of the firm can be finally put in the form :

> **Find x_1 and x_2 such that the profit $P = 2x_1 + 3x_2$ is maximum,**
> **Subject to the conditions :**
> $x_1 + x_2 \le 400, \quad 2x_1 + x_2 \le 600, \quad x_1 \ge 0, \ x_2 \ge 0.$

Example 2. *A company produces two types of Hats. Each hat of the first type requires twice as much labour time as the second type. If all hats are of the second type only, the company can produce a total of 500 hats a day. The market limits daily sales of the first and second type to 150 and 250 hats. Assuming that the profits per hat are Rs. 8 for type A and Rs. 5 for type B, formulate the problem as a linear programming model in order to determine the number of hats to be produced of each type so as to maximize the profit.*

Formulation. Let the company produce x_1 hats of type *A* and x_2 hats of type *B* each day. So the profit P after selling these two products is given by the linear function :

$$P = 8x_1 + 5x_2 \quad \textit{(objective function)}$$

Since the company can produce at the most 500 hats in a day and *A* type of hats require twice as much time as that of type *B*, production restriction is given by $2tx_1 + tx_2 \le 500\, t$, where *t* is the labour time per unit of second type, *i.e.*

$$2x_1 + x_2 \le 500 .$$

But, there are limitations on the sale of hats, therefore further restrictions are :

$$x_1 \le 150, \quad x_2 \le 250 .$$

Also, since the company cannot produce negative quantities,

$$x_1 \ge 0 \quad \text{and} \quad x_2 \ge 0 .$$

Hence the problem can be finally put in the form :

> **Find x_1 and x_2 such that the profit $P = 8x_1 + 5x_2$ is maximum,**
> **Subject to the restrictions :**
> $2x_1 + x_2 \le 500, x_1 \le 150, x_2 \le 250, x_1 \ge 0, \ x_2 \ge 0.$

*Here the constraint $2x_1 + x_2 = 600$ is not justified because using machine H for less than 10 hrs (if possible) will be more profitable.

Example 3. *The manufacturer of patent medicines is proposed to prepare a production plan for medicines A and B. There are sufficient ingredient available to make 20,000 bottles of medicine A and 40,000 bottles of medicine B, but there are only 45,000 bottles into which either of the medicines can be filled. Further, it takes three hours to prepare enough material to fill 1000 bottles of medicine A and one hour to prepare enough material to fill 1000 bottles of medicine B, and there are 66 hours available for this operation. The profit is Rs. 8 per bottle for medicine A and Rs. 7 per bottle for medicine B.*
(i) *Formulate this problem as a L.P.P.*
(ii) *How the manufacturer schedule his production in order to maximize profit.*

Formulation. (i) Suppose the manufacturer produces x_1 and x_2 thousand of bottles of medicines A and B, respectively. Since it takes three hours to prepare 1000 bottles of medicine A, the time required to fill x_1 thousand bottles of medicine A will be $3x_1$ hours. Similarly, the time required to prepare x_2 thousand bottles of medicine B will be x_2 hours. Therefore, total time required to prepare x_1 thousand bottles of medicine A and x_2 thousand bottles of medicine B will be $3x_1 + x_2$ hours.

Now since the total time available for this operation is 66 hours, $3x_1 + x_2 \leq 66$.

Since there are only 45 thousand bottles available for filling medicines A and B, $x_1 + x_2 \leq 45$.

There are sufficient ingredients available to make 20 thousand bottles of medicine A and 40 thousand bottles of medicine B, hence $x_1 \leq 20$ and $x_2 \leq 40$.

Number of bottles being non-negative, $x_1 \geq 0$, $x_2 \geq 0$.

At the rate of Rs. 8 per bottle for type A medicine and Rs. 7 per bottle for type B medicine, the total profit on x_1 thousand bottles of medicine A and x_2 thousand bottles of medicine B will become

$$P = 8 \times 1000\, x_1 + 7 \times 1000\, x_2 \qquad \text{or} \qquad P = 8000\, x_1 + 7000\, x_2 .$$

Thus, the linear programming problem is :
Max. $P = 8000\, x_1 + 7000\, x_2$ **, subject to the constraints :**

$$3x_1 + x_2 \leq 66, x_1 + x_2 \leq 45, x_1 \leq 20, x_2 \leq 40$$

and $x_1 \geq 0, x_2 \geq 0$.

(ii) See ***Example* 28** (page 76) for its solution by graphical method.

Example 4. *A toy company manufactures two types of doll, a basic version—doll A and a delux version—doll B. Each doll of type B takes twice as long to produce as one of type A, and the company would have time to make a maximum of 2000 per day. The supply of plastic is sufficient to produce 1500 dolls per day (both A and B combined). The deluxe version requires a fancy dress of which there are only 600 per day available. If the company makes a profit of Rs. 3.00 and Rs. 5.00 per doll, respectively on doll A and B, then how many of each doll should be produced per day in order to maximize the total profit. Formulate this problem.* **[Kanpur B.Sc. 90; Meerut 90]**

Formulation. Let x_1 and x_2 be the number of dolls produced per day of type A and B, respectively. Let the doll A require t hrs so that the doll B require $2t$ hrs. So the total time to manufacture x_1 and x_2 dolls should not exceed $2{,}000t$ hrs. Therefore, $tx_1 + 2tx_2 \leq 2000t$. Other constraints are simple. Then the linear programming problem becomes :

Maximize $P = 3x_1 + 5x_2$
Subject to the restrictions :

$$x_1 + 2x_2 \leq 2000 \quad \text{(time constraint)}$$
$$x_1 + x_2 \leq 1500 \quad \text{(plastic constraint)}$$
$$x_2 \leq 600 \quad \text{(dress constraint)}$$

and non-negativity restrictions

$$x_1 \geq 0, x_2 \geq 0 .$$

Note : See ***Example* 26** (page 74) for its solution by graphical method.

Example 5. *In a chemical industry, two products A and B are made involving two operations. The production of B also results in a by-product C. The product A can be sold at Rs. 3 profit per unit and B at Rs. 8 profit per unit. The by-product C has a profit of Rs. 2 per unit, but it cannot be sold as the destruction cost is Re. 1 per unit. Forecasts show that up to 5 units of C can be sold. The company gets 3 units of C for each unit of A and B produced. Forecasts show that they can sell all the units of A and B produced. The manufacturing times are 3 hours per unit for A on operation one and two respectively and 4 hours and 5 hours per unit for B on operation one and two respectively. Because the product C results from producing B, no time is used in producing C. The available times are 18 and 21 hours of operation one and two respectively. The company question : how much A and B should be produced keeping C in mind to make the highest profit. Formulate LP model for this problem.*

Formulation. Let x_1 , x_2 , x_3 be the number of units produced of product A , B , C respectively. Then the profit gained by the industry is given by $\mathbf{P} = 3x_1 + 8x_2 + 2x_3$.

Here it is assumed that all the units of product A and B are sold.

In first operation, A takes 3 hours of manufacturer's time and B takes 4 hours of manufacturer's time, therefore total number of hours required in first operation becomes $3x_1 + 4x_2$.

In second operation, A takes 3 hours of manufacturer's time and B takes 5 hours of manufacturer's time, therefore the total number of hours used in second operation becomes $3x_1 + 5x_2$.

Since there are 18 hours available in first operation and 21 hours in second operation, the restrictions become : $3x_1 + 4x_2 \leq 18$, $3x_1 + 5x_2 \leq 21$.

Also, the company gets 3 units of by-product C for each unit of B produced, therefore the total number of units of product B and C produced becomes : $x_2 + 3x_3$.

But, the maximum number of units of C can be sold is 5, therefore $x_2 + 3x_3 \leq 5$.

Thus the allocation problem of the industry can be finally put in the form :

Find the value of x_1 , x_2 , x_3 so as to maximize

$\mathbf{P} = 3x_1 + 8x_2 + 2x_3$ subject to the restrictions :

$$3x_1 + 4x_2 \leq 18$$

$$3x_1 + 5x_2 \leq 21$$

$$x_2 + 3x_3 \leq 5,$$

with non-negativity conditions : $x_1 \geq 0 , x_2 \geq 0 , x_3 \geq 0$.

Example 6. *A firm can produce three types of cloth, say : A, B, and C. Three kinds of wool are required for it, say : red, green and blue wool. One unit length of type A cloth needs 2 meters of red wool and 3 meters of blue wool ; one unit length of type B cloth needs 3 meters of red wool, 2 meters of green wool and 2 meters of blue wool ; and one unit of type C cloth needs 5 meters of green wool and 4 meters of blue wool. The firm has only a stock of 8 meters of red wool, 10 meters of green wool and 15 meters of blue wool. It is assumed that the income obtained from one unit length of type A cloth is Rs. 3.00, of type B cloth is Rs. 5.00, and of type C cloth is Rs. 4.00.*

Determine, how the firm should use the available material so as to maximize the income from the finished cloth.

Formulation. It is often convenient to construct the *Table 4.2* after understanding the problem carefully.

Table 4.2

Quality of wool	*Type of Cloth*			*Total quantity of wool available (in meters)*
	A (x_1)	B (x_2)	C (x_3)	
Red	2	3	0	8
Green	0	2	5	10
Blue	3	2	4	15
Income per unit length of cloth	Rs. 3.00	Rs. 5.00	Rs. 4.00	

Let x_1, x_2 and x_3 be the quantity (in meters) produced of cloth type A, B, C respectively. Since 2 meters of red wool are required for each meter of cloth A and x_1 meters of this type of cloth are produced, so $2x_1$ meters of red wool will be required for cloth A.

Similarly, cloth B requires $3x_2$ meters of red wool and cloth C does not require red wool. Thus, total quantity of red wool becomes :

$$2x_1 + 3x_2 + 0x_3 \text{ (red wool)}$$

Following similar arguments for *green* and *blue* wool,

$$0x_1 + 2x_2 + 5x_3 \text{ (green wool)}$$

$$3x_1 + 2x_2 + 4x_3 \text{ (blue wool)}$$

Since not more than 8 meters of red, 10 meters of green and 15 meters of blue wool are available, the variables x_1, x_2, x_3 must satisfy the following restrictions :

$$\begin{aligned} 2x_1 + 3x_2 \qquad &\le 8 \\ 2x_2 + 5x_3 &\le 10 \\ 3x_1 + 2x_2 + 4x_3 &\le 15\,. \end{aligned} \qquad \text{...(4.1)}$$

Also, negative quantities cannot be produced. Hence x_1, x_2, x_3 must satisfy the non-negativity restrictions :

$$x_1 \ge 0\,, x_2 \ge 0\,, x_3 \ge 0. \qquad \text{...(4.2)}$$

The total income from the finished cloth is given by

$$\text{P} = 3x_1 + 5x_2 + 4x_3\,. \qquad \text{...(4.3)}$$

Thus the problem now becomes to find x_1, x_2, x_3 satisfying the restrictions (4·1) and (4·2) and maximizing the profit function P.

Note. This linear programming problem has been solved by simplex method as *Example* 5 (page 125).

Example 7. *A firm manufactures 3 products A, B and C. The profits are Rs. 3, Rs. 2 and Rs. 4 respectively. The firm has 2 machines and below is the required processing time in minutes for each machine on each product.*

		Product		
		A	B	C
Machine	G	4	3	5
	H	2	2	4

Machine G and H have 2,000 and 2,500 machine-minutes, respectively. The firm must manufacture 100 A's, 200 B's and 50 C's, but no more than 150 A's.

Setup an L.P. problem to maximize profit. Do not solve it.

Formulation. Let x_1, x_2, x_3 be the number of products A, B and C, respectively.

Since the profits are Rs. 3, Rs. 2 and Rs. 4 respectively, the total profit gained by the firm after selling these three products is given by $\mathbf{P} = 3x_1 + 2x_2 + 4x_3$.

Now the total number of minutes required in producing these three products at machine *G and H are given by*

$$4x_1 + 3x_2 + 5x_3\,, \quad \text{and} \quad 2x_1 + 2x_2 + 4x_3, \text{ respectively.}$$

But, there are only 2,000 minutes available at machine G and 2,500 minutes at machine H, therefore the restrictions will be

$$4x_1 + 3x_2 + 5x_3 \le 2{,}000 \quad \text{and} \quad 2x_1 + 2x_2 + 4x_3 \le 2{,}500\,.$$

Also, since the firm manufactures 100 A's, 200 B's and 50 C's but not more than 150 A's, therefore further restrictions become :

$$100 \le x_1 \le 150,\ 200 \le x_2 \ge 0 \text{ and } 50 \le x_3 \ge 0\,.$$

Hence the allocation problem of the firm can be finally put in the form :

Find the value of x_1, x_2, x_3 so as to maximize

$$P = 3x_1 + 2x_2 + 4x_3$$

subject to the constraints :

$$4x_1 + 3x_2 + 5x_3 \le 2{,}000$$
$$2x_1 + 2x_2 + 4x_3 \le 2{,}500$$
$$100 \le x_1 \le 150,\ 200 \le x_2 \ge 0,\ 50 \le x_3 \ge 0.$$

Example 8. *A farmer has 100 acre farm. He can sell all tomatoes, lettuce, or radishes he can raise. The price he can obtain is Re. 1.00 per kg for tomatoes, Rs. 0·75 a head for lettuce and Rs. 2·00 per kg for radishes. The average yield per-acre is 2,000 kg of tomatoes, 3000 heads of lettuce, and 1000 kgs of radishes. Fertilizer is available at Rs. 0·50 per kg and the amount required per acre is 100 kgs each for tomatoes and lettuce, and 50 kgs for radishes. Labour required for sowing, cultivating and harvesting per acre is 5 man-days for tomatoes and radishes, and 6 man-days for lettuce. A total of 400 man-days of labour are available at Rs. 20·00 per man-day.*

Formulate this problem as a linear programming model to maximize the farmer's total profit.

[Kanpur (B.Sc.) 93, 92]

Formulation. Farmer's problem is to decide how much area should be alloted to each type of crop he wants to grow to maximize his total profit. Let the farmer decide to allot x_1, x_2 and x_3 acre of his land to grow tomatoes, lettuce and radishes respectively. So the farmer will produce $2000x_1$ kgs of tomatoes, $3000x_2$ heads of lettuce, and $1000x_3$ kgs of radishes.

Therefore, total sale will be $= \text{Rs. } [2000\,x_1 + 0{\cdot}75 \times 3000\,x_2 + 2 \times 1000\,x_3]$

Fertilizer expenditure will be $= \text{Rs. } [0{\cdot}50\,\{100\,(x_1 + x_2) + 50x_3\}]$

Labour expenditure will be $= \text{Rs. } [20 \times (5x_1 + 6x_2 + 5x_3)]$

Therefore, farmer's net profit will be

$$P = \text{Total sale (in Rs.)} - \text{Total expenditure (in Rs.)}$$

or $P = [2000\,x_1 + 0{\cdot}75 \times 3000\,x_2 + 2 \times 1000\,x_3] - 0{\cdot}50 \times [100\,(x_1 + x_2) + 50\,x_3] - 20 \times [5x_1 + 6x_2 + 5x_3]$

or $P = 1850\,x_1 + 2080\,x_2 + 1875\,x_3$.

Since total area of the farm is restricted to 100 acre, $x_1 + x_2 + x_3 \le 100$.

Also, the total man-days labour is restricted to 400 man-days, therefore, $5x_1 + 6x_2 + 5x_3 \le 400$.

Hence the farmer's allocation problem can be finally put in the form :

Find the value of x_1, x_2, x_3 so as to maximize :

$$P = 1850\,x_1 + 2080\,x_2 + 1875\,x_3,$$

Subject to the conditions :

$$x_1 + x_2 + x_3 \le 100,$$
$$5x_1 + 6x_2 + 5x_3 \le 400,$$
and $$x_1, x_2, x_3 \ge 0.$$

Example 9. *A manufacturer produces three models (I, II and III) of a certain product. He uses two types of raw material (A and B) of which 4000 and 6000 units repectively are available. The raw material requirements per unit of the three models are given below :*

Raw Material	*Requirement per unit of given model*		
	I	*II*	*III*
A	2	3	5
B	4	2	7

The labour time for each unit of model I is twice that of model II and three times that of model III. The entire labour force of the factory can produce the equivalent of 2500 units of model I. A market survey indicates that the minimum demand of the three models are 500, 500 and 375 units respectively. However, the ratios of the number of units produced must be equal to 3 : 2 : 5. Assume that the profit per unit of models I, II *and III are rupees 60, 40 and 100 respectively. Formulate the problem as a linear programming model in order to determine the number of units of each product which will maximize profit.* **[JNTU (B.Tech) 98]**

Formulation. Let the manufacturer produce x_1, x_2, x_3 units of model I, II and III, respectively. Then, the raw material constraints will be

$$2x_1 + 3x_2 + 5x_3 \leq 4{,}000 \quad \text{(for A)}$$
$$4x_1 + 2x_2 + 7x_3 \leq 6{,}000 \quad \text{(for B)}$$

Suppose it takes labour time t for producing one unit of model I, so by the given condition it will take $t/2$ and $t/3$ labour time for producing one unit of model II and III, respectively.

As the factory can produce 2500 units of model I, so the restriction on the production time will be $tx_1 + (t/2)x_2 + (t/3)x_3 \leq 2500t$, *i.e.*,

$$x_1 + 1/2\, x_2 + 1/3\, x_3 \leq 2500.$$

Also, since at least 500 units of model type I and II each and 375 units of model III are demanded, the constraints of market demand needs,

$$x_1 \geq 500, \quad x_2 \geq 500, \quad \text{and} \quad x_3 \geq 375.$$

But, the ratio of the number of units of different types of models is 3 : 2 : 5, we have $1/3\, x_1 = 1/2\, x_2$ and $1/2\, x_2 = 1/5\, x_3$.

Since the profit per unit on model I, II and III are Rs. 60, Rs. 40 and Rs. 100 respectively, the objective function is to maximize the profit : $P = 60x_1 + 40x_2 + 100x_3$.

Thus, the linear programming problem is :
To maximize : $P = 60x_1 + 40x_2 + 100x_3$,
Subject to the constraints :

$$2x_1 + 3x_2 + 5x_3 \leq 4{,}000$$
$$4x_1 + 2x_2 + 7x_3 \leq 6{,}000$$
$$x_1 + \frac{1}{2}x_2 + \frac{1}{3}x_3 \leq 2{,}500$$
$$\frac{1}{3}x_1 = \frac{1}{2}x_2, \frac{1}{2}x_2 = \frac{1}{5}x_3$$

and $x_1 \geq 500, x_2 \geq 500, x_3 \geq 375.$

Example 10. (Diet Problem.) *One of the interesting problems in linear programming is that of balanced diet. Dieticians tell us that a balanced diet must contain quantities of nutrients such as calories, minerals, vitamins, etc. Suppose that we are asked to find out the food that should be recommended from a large number of alternative sources of these nutrients so that the total cost of food satisfying minimum requirements of balanced diet is the lowest.*

The medical experts and dieticians tell us that it is necessary for an adult to consume at least 75 g of proteins, 85 g of fats, and 300 g of carbohydrates daily. The following table gives the food items (which are readily available in the market), analysis, and their respective costs.

Food type	Food value (gms.) per 100g			Cost per kg. (Rs.)
	Proteins	Fats	Carbohydrates	
1	8·0	1.5	35.0	1.00
2	18.0	15.0	—	3.00
3	16.0	4.0	7.0	4.00
4	4.0	20.0	2.5	2.00
5	5.0	8.0	40.0	1.50
6	2.5	—	25.0	3.00
Minimum daily requirements	75	85	300	

Formulation. Let x_1, x_2, x_3, x_4, x_5 and x_6 units of food types respectively be used per day in a diet and the total diet must at least supply the minimum requirements. The object is to minimize total cost C of diet. The objective function thus becomes

$$Z = x_1 + 3x_2 + 4x_3 + 2x_4 + 1{\cdot}5\, x_5 + 3x_6.$$

Since 8, 18, 16, 4, 5 and 2.5 gms of proteins are available from 100 gm unit of each type of food respectively, total proteins available from x_1, x_2, x_3, x_4, x_5 and x_6 units of each food respectively will be $8x_1 + 18x_2 + 16x_3 + 4x_4 + 5x_5 + 2.5x_6$ gms. daily.

But minimum daily requirement of proteins as prescribed is 75 gms. Hence, the *protein* requirement constraint is

$$8x_1 + 18x_2 + 16x_3 + 4x_4 + 5x_5 + 2{\cdot}5x_6 \geq 75 \; (Protein)$$

Similarly, *fats* and *carbohydrates* requirement constraints are obtained respectively as given below :

$$1.5x_1 + 15x_2 + 4x_3 + 20x_4 + 8x_5 + 0x_6 \geq 85 \; (Fats)$$

$$35x_1 + 0x_2 + 7x_3 + 2.5x_4 + 40x_5 + 25x_6 \geq 300 \; (Carbohydrates)$$

Further, $x_1, x_2, x_3\; x_4, x_5, x_6$ are all non-negative quantities, *i.e.*

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0, x_6 \geq 0.$$

Generalization. A generalization of this problem is as follows :

Let a_{ij} be the number of units of nutrient i in one unit of food j, x_j be the units of food j used per day, and b_i be the requirement of the ith nutrient. Thus, the objective function becomes :

Minimize $Z = c_1 x_1 + c_2 x_2 + \ldots c_n x_n$ subject to the nutrient requirements :

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \geq b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \geq b_2$$
$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \geq b_m$$
$$x_1, x_2, x_3, \ldots, x_n \geq 0,$$

where n defines the number of food items, m the different nutrients, and c_j the cost per unit of food $j\; (j = 1, 2, \ldots, n)$.

It should be noted that diet problem as formulated above does not provide variation in the diet. The same combination must be taken every day (to keep ones health good). It is assumed that the diet suggested is palatable (let us hope so). The constraints place the bound on the minimum size of the food value and it may turnout that a person may be taking too much of carbohydrate while trying to keep the protein requirements (too bad for fattys). Hence an upper bound would also be desirable if the intake beyond limit is bad for health.

Example 11. *A manufacturer of biscuits is considering four types of gift packs containing three types of biscuits : orange cream (OC), chocolate cream (CC), and wafers (W). Market research conducted recently to assess the preferences of the consumers shows the following types of assortments to be in good demand :*

Assortments	Contents	Selling price per kg. (Rs.)
A	Not less than 40% of OC, not more than 20% of CC, any quantity of W	20
B	Not less than 20% of OC, not more than 40% of CC, any quantity of W	25
C	Not less than 50% of OC, not more than 10% of CC, any quantity of W	22
D	No restrictions	12

For the biscuits, the manufacturing capacity and costs are given below :

Biscuits variety :	*OC*	*CC*	*W*
Plant capacity (kg/day) :	200	200	150
Manufacturing cost (Rs./kg) :	8	9	7

Formulate a linear programming model to find the production schedule which maximizes the profit assuming that there are no market restrictions.

Formulation. Let the decision variables $x_{ij}\; (i = A, B, C, D; j = 1, 2, 3)$ be defined as follows :

(i) For the gift pack A, x_{A1}, x_{A2}, x_{A3} denote the quantity in kg. of OC, CC, and W type of biscuits;

(ii) For the gift pack B, $x_{B1}\; x_{B2}, x_{B3}$ denote the quantity in kg. of OC, CC, and W type of biscuits;

(iii) For the gift pack C, x_{C1}, x_{C2}, x_{C3} denote the quantity in kg. of OC, CC, and W type of biscuits;

(iv) For the gift pack D, x_{D1}, x_{D2}, x_{D3} denote the quantity in kg. of OC, CC, and W type of biscuits;

Now the given data can be put in the form of linear programming problem as follows :

Maximize, $P = 20\,(x_{A1}+x_{A2}+x_{A3}) + 25\,(x_{B1}+x_{B2}+x_{B3}) + 22\,(x_{C1}+x_{C2}+x_{C3}) + 12\,(x_{D1}+x_{D2}+x_{D3})$

$$- 8\,(x_{A1}+x_{B1}+x_{C1}+x_{D1}) - 9\,(x_{A2}+x_{B2}+x_{C2}+x_{D2}) - 7\,(x_{A3}+x_{B3}+x_{C3}+x_{D3})$$

$$= 12\,x_{A1} + 11x_{A2} + 13x_{A3} + 17x_{B1} + 16x_{B2} + 18x_{B3} + 14x_{C1} + 13x_{C2} + 15x_{C3} + 4x_{D1} + 3x_{D2} + 5x_{D3};$$

subject to the constraints :

Gift pack A : $x_{A1} \ge 0.40\,(x_{A1}+x_{A2}+x_{A3}),\ x_{A2} \le 0.20\,(x_{A1}+x_{A2}+x_{A3})$

Gift pack B : $x_{B1} \ge 0.20\,(x_{B1}+x_{B2}+x_{B3}),\ x_{B2} \le 0.40\,(x_{B1}+x_{B2}+x_{B3})$

Gift pack C : $x_{C1} \ge 0.50\,(x_{C1}+x_{C2}+x_{C3}),\ x_{C2} \le 0.10\,(x_{C1}+x_{C2}+x_{C3})$

Plant capacity constraints are :

$$x_{A1}+x_{B1}+x_{C1}+x_{D1} \le 200,\ x_{A2}+x_{B2}+x_{C2}+x_{D2} \le 200,\ x_{A3}+x_{B3}+x_{C3}+x_{D3} \le 150$$

$$x_{ij} \ge 0 \text{ (for } i = A, B, C, D \text{ and } j = 1, 2, 3).$$

Example 12. *A complete unit of a certain product consists of four units of component A and three units of component B. Two components (A and B) are manufactured from two different raw materials of which 100 units and 200 units, respectively, are available. Three departments are engaged in the production process with each department using a different method for manufacturing the components. The following table gives the raw material requirements per production run and the resulting units of each component. The objective is to determine the number of production runs for each department which will maximize the total number of component units of the final product.*

Department	*Input per run (units)*		*Out put per run (units)*	
	Raw material I	*Raw material II*	*Component A*	*Component B*
1	7	5	6	4
2	4	8	5	8
3	2	7	7	3

Formulation. Let x_1, x_2, x_3 be the number of production runs for the departments 1,2,3 respectively.

The total number of units produced by three departments :

$$6x_1 + 5x_2 + 7x_3 \text{ (component } A), \qquad 4x_1 + 8x_2 + 3x_3 \text{ (component } B)$$

The restrictions on the raw materials I and II are, respectively, given by

$$7x_1 + 4x_2 + 2x_3 \le 100 \quad \text{and} \quad 5x_1 + 8x_2 + 7x_3 \le 200.$$

Since the objective function is to maximize the total number of units of the final product and each such unit requires 4 units of component A and 3 units of component B, the maximum number of units of the final product cannot exceed the smaller value of

$$\tfrac{1}{4}(6x_1 + 5x_2 + 7x_3) \quad \text{and} \quad \tfrac{1}{3}(4x_1 + 8x_2 + 3x_3).$$

The objective function thus becomes :

Maximize $z = \min\,[\tfrac{1}{4}(6x_1 + 5x_2 + 7x_3),\ \tfrac{1}{3}(4x_1 + 8x_2 + 3x_3)]$

Since this objective function is not linear, a suitable transformation can be used to reduce the above model to an acceptable linear programming format.

Suppose, $\min.\,[\tfrac{1}{4}(6x_1 + 5x_2 + 7x_3),\ \tfrac{1}{3}(4x_1 + 8x_2 + 3x_3)] = v.$

Therefore, $\tfrac{1}{4}(6x_1 + 5x_2 + 7x_3) \ge v$ and $\tfrac{1}{3}(4x_1 + 8x_2 + 3x_3) \ge v.$

In fact, at least one of these two inequalities must hold as an equation in any solution because the number of final assembly units, v, is maximized. Then, its upper limit is specified by the smaller of the left hand sides of above two inequalities. This indicates that the two inequalities are equivalent to the original equation defining v.

Now the above problem can be put into the following linear programming form : maximize $z = v$, subject to the constraints :

$$6x_1 + 5x_2 + 7x_3 - 4v \ge 0, \quad 4x_1 + 8x_2 + 3x_3 - 3v \ge 0,$$

$$7x_1 + 4x_2 + 2x_3 \le 100, \quad 5x_1 + 8x_2 + 7x_3 \le 200, \quad \text{and} \quad x_1, x_2, x_3, v \ge 0.$$

Example 13. *A leading C.A. is attempting to determine a 'best' investment portfolio and is considering six alternative investment proposals. The following table indicates point estimates for the price per share, the annual growth rate in the price per share, the annual dividend per share and a measure of the risk associated with each investment.*

Portfolio Data

Shares under consideration :	A	B	C	D	E	F
Current price per share (Rs.)	80	100	160	120	150	200
Projected annual growth rate	0·08	0·07	0·10	0·12	0·09	0·15
Projected annual dividend per share (Rs.)	4·00	4·50	7·50	5·50	5·75	0·00
Projected risk in return	0·05	0·03	0·10	0·20	0·06	0·08

The total amount available for investment is Rs. 25 lakhs and the following conditions are required to be satisfied.

(i) The maximum rupee amount to be invested in alternative F is Rs. 2,50,000.

(ii) No more than Rs. 5,00,000 should be invested in alternatives A and B combined.

(iii) Total weighted risk should not be greater than 0·10, where

$$\text{Total weighted risk} = \frac{\text{(Amount invested in alternative j) (Risk of alternative j)}}{\text{Total amount invested in all the alternatives}}$$

(iv) For the sake of diversity, at least 100 shares of each stock should be purchased.

(v) At least 10 per cent of the total investment should be in alternatives A and B combined.

(vi) Dividends for the year should be at least 10,000.

Rupees return per share of stock is defined as price per share one year hence less current price per share PLUS dividend per share. If the objective is to maximize total rupee return, formulate the linear programming model for determining the optimal number of shares to be purchased in each of the shares under consideration. You may assume that the time horizon for the investment is one year. The formulated LP problem is not required to be solved. **[C.A. (Final) Nov. 91]**

Formulation. Let x_1, x_2, x_3, x_4, x_5 and x_6 represent the number of shares to be purchased in each of the six investment proposals A, B, C, D, E and F.

Rupee return per share= Price per share one year hence – current price per share + dividend per share)
= Current price per share × Projected annual growth rate
(*i.e.* Project growth each year)+ dividend per share

Thus, we compute the following data :

Investment Alternatives	:	A	B	C	D	E	F
No. of shares purchased	:	x_1	x_2	x_3	x_4	x_5	x_6
Projected growth for each share (Rs.)	:	6·40	7·00	16·00	14·40	13·50	30·00
Projected annual dividend per share (Rs.)	:	4·00	4·50	7·50	5·50	5·75	0·00
Rupee return per share	:	10·40	11·50	23·50	19·90	19·25	30·00

The Chartered Accountant wants to maximize the total rupee return, thus the objective function of the linear programming problem is given by :

$$\text{Maximize } R = 10{\cdot}40x_1 + 11{\cdot}50x_2 + 23{\cdot}50x_3 + 19{\cdot}90x_4 + 19{\cdot}25x_5 + 30{\cdot}00x_6,$$

subject to the constraints (1) to (7) as stated below.

Since the total amount available for investment is Rs. 25 lakhs, therefore

(1) $80x_1 + 100x_2 + 160x_3 + 120x_4 + 150x_5 + 200x_6 \le 25{,}00{,}000$

(2) $200x_6 \le 2{,}50{,}000$ [from condition (1)]

(3) $80x_1 + 100x_2 \le 5{,}00{,}000$ [from condition (2)]

(4) According to condition (3) of the problem

$$\left[\frac{80x_1\,(0{\cdot}05) + 100x_2\,(0{\cdot}03) + 160x_3\,(0{\cdot}10) + 120x_4(0{\cdot}20) + 150x_5\,(0{\cdot}06) + 200x_6(0{\cdot}08)}{80x_1 + 100x_2 + 160x_3 + 120x_4 + 150x_5 + 200x_6}\right] \le 0{\cdot}10$$

or $\quad 4x_1 + 3x_2 + 16x_3 + 24x_4 + 9x_5 + 16x_6 \le 8x_1 + 10x_2 + 16x_3 + 12x_4 + 15x_5 + 20x_6$

or $\quad -4x_1 - 7x_2 + 0x_3 + 12x_4 - 6x_5 - 4x_6 \le 0$

(5) $x_1 \ge 100, x_2 \ge 100, x_3 \ge 100, x_4 \ge 100, x_5 \ge 100, x_6 \ge 100$ [from condition (4)]

(6) $80x_1 + 100x_2 \ge 0{\cdot}10\,(80x_1 + 100x_2 + 160x_3 + 120x_4 + 150x_5 + 200x_6)$ [from condition (5)]

or $\quad 80x_1 + 100x_2 \ge 8x_1 + 10x_2 + 16x_3 + 12x_4 + 15x_5 + 20x_6$

or $\quad 72x_1 + 90x_2 - 16x_3 - 12x_4 - 15x_5 - 20x_6 \ge 0$

(7) $4x_1 + 4{\cdot}5x_2 + 7{\cdot}5x_3 + 5{\cdot}5x_4 + 5{\cdot}75x_5 \ge 10{,}000$ [from condition (6)]

Finally, combining all the constraints from (1) to (7), the desired linear programming problem is formulated.

Example 14. *Consider a company that must produce two products over a production period of three months of duration. The company can pay for materials and labour from two sources : company funds and borrowed funds.*

The firm faces three decisions :

(1) How many units should it produce of Product 1 ?

(2) How many units should it produce of Product 2 ?

(3) How much money should it borrow to support the production of the two products ?

In making these decisions, the firm wishes to maximize the profit contribution subject to the conditions stated below :

(i) Since the company's products are enjoying a seller's market, it can sell as many units as it can produce. The company would therefore like to produce as many units as possible subject to production capacity and financial constraints. The capacity constraints, together with cost and price data, are given in Table–1.

Table–1 : Capacity, Price and Cost data

Product	Selling Price (Rs. per unit)	Cost of Productionj (Rs. per unit)	Required Hours per unit in Department		
			A	B	C
1	14	10	0·5	0·3	0·2
2	11	8	0·3	0·4	0·1
Available hours per production period of three months :			500	400	200

(ii) The available company funds during the production period will be Rs. 3 lakhs.

(iii) A bank will give loans upto Rs. 2 lakhs per production period at an interest rate of 20 per cent per annum provided the company's acid (quick) test ratio is at least 1 to 1 while the loan is outstanding. Take a simplified acid-test ratio given by

$$\frac{\textit{Surplus cash on hand after production + Accounts receivable}}{\textit{Bank borrowings + Interest occurred there on}}$$

(iv) Also make sure that the needed funds are made available for meeting the production costs.

Formulate the above as a linear programming problem. **[C.A. (Nov. 92)]**

Formulation. Let x_1 = no. of units of product 1 produced, x_2 = no. of units of product 2 produced, and x_3 = amount of money borrowed.

The profit contribution per unit of each product is given by (selling price – variable cost of production). Total profit can be computed by [summing of the profits from producing the two products – the cost associated with borrowed funds (if any)]. The objective function is thus obtained as :

Maximize $\quad \text{P} = (14 - 10)\,x_1 + (11 - 8)\,x_2 - 0{\cdot}05\,x_3 = 4x_1 + 3x_2 - 0{\cdot}05x_3$

(since the interest rate is 20% per annum, hence it will be 5% for a period of three months.)

Subject to the following constraints :

The production capacity constraints for each department (as given by table 1) are :

$0{\cdot}5x_1 + 0{\cdot}3x_2 \le 500$ ...(1) $\quad 0{\cdot}3x_1 + 0{\cdot}4x_2 \le 400$ ...(2) $\quad 0{\cdot}2x_1 + 0{\cdot}1x_2 \le 200$ ...(3)

The funds for available production include both Rs. 3,00,000 cash that the firm possesses and any borrowed funds maximum up to Rs. 2,00,000. Consequently, production is limited to the extent that funds are available to pay for production costs. The constraint representing this relationship is given by :

Funds required for production ≤ Funds available

i.e. $10x_1 + 8x_2 \le \text{Rs. } 3{,}00{,}000 + x_3$ or $10x_1 + 8x_2 - x_3 \le \text{Rs. } 3{,}00{,}000$ ...(4)

The borrowed funds constraint [from condition (iii) of the problem] is given by $x_3 \le \text{Rs. } 2{,}00{,}000$

The constraint based on the acid-test condition is developed as follows :

$$\frac{\text{Surplus cash on hand after production + Accounts receivable}}{\text{Bank borrowings + Interest occured there on}} \ge 1$$

i.e.
$$\frac{(3{,}00{,}000 + x_3 - 10x_1 - 8x_2) + 14x_1 + 11x_2}{x_3 + 0{\cdot}05x_3} \ge 1$$

or
$$3{,}00{,}000 + x_3 + 4x_1 + 3x_2 \ge (x_3 + 0{\cdot}05x_3)$$

or
$$-4x_1 - 3x_2 + 0{\cdot}05x_3 \le 3{,}00{,}000 \quad ...(5)$$

Finally the linear programming problem is formulated as :

Max. $P = 4x_1 + 3x_2 - 0{\cdot}05x_3$, subject to the constraints :

$0{\cdot}5x_1 + 0{\cdot}3x_2 \le 500$, $0{\cdot}3x_1 + 0{\cdot}4x_2 \le 400$, $0{\cdot}2x_1 + 0{\cdot}1x_2 \le 200$

$10x_1 + 8x_2 - x_3 \le \text{Rs. } 3{,}00{,}000$, $x_3 \le \text{Rs. } 2{,}00{,}000$

$-4x_1 - 3x_2 + 0{\cdot}05x_3 \le \text{Rs. } 3{,}00{,}000$, where $x_1, x_2, x_3 \ge 0$.

Example 15. *WELLTYPE Manufacturing company produces three types of typewriters. All the three models are required to be machined first then assembled. The time required for various models are as follows :*

Types	Manual typewriters	Electronic typewriters	Deluxe elec. typewriters
Machine time (in hours)	15	12	14
Assembly time (in hours)	4	3	5

The total available machine time and assembly time are 3000 hours and 1,200 hours, respectively. The data regarding the selling price and variable costs for the three types are :

Type	Manual	Electronic	Deluxe Electronic
Selling price (Rs.)	4,100	7,500	14,600
Labour, material and other variable costs (Rs.)	2,500	4,500	9,000

The company sells all the three types on credit basis, but will collect the amounts on the first of next month. The labour, material and other variable expenses will have to be paid in cash. This company has taken a loan of Rs. 40,000 from a co-operative bank and this company will have to repay it to the bank on Ist April, 1993. The TNC Bank from whom this company has borrowed Rs. 60,000 has expressed its approval to renew the loan.

The Balance Sheet of this company as on 31.3.93 is as follows :

Liabilities	Rs.	Assets	Rs.
Equity share capital	1,50,000	Land	90,000
Capital reserve	15,000	Building	70,000
General Reserve	1,10,000	Plant & Machinery	1,00,000
Profit & Loss a/c	25,000	Furniture & Fixures	15,000
Long term loan	1,00,000	Vehicles	30,000
Loan from TNC Bank	60,000	Inventory	5,000
Loan from Co-op. Bank	40,000	Receivables	50,000
		Cash	1,40,000
Total	5,00,000	Total	5,00,000

The company will have to pay a sum of Rs. 10,000 towards the salary from top management executives and other fixed overheads for the month. Interest on long term loans is to be paid every month at 24% per annum. Interest on loans from TNC and Co-operative Banks may be taken to be Rs. 1,200 for the month. Also this company has promised to deliver 2 Manual typewriters and 8 Deluxe-Electronic type writers to one of its valued customers next month. Also make sure that the level of operations in this company is subject to the availability of cash next month. This company will also be able to sell all their types of type writers in the market. The senior manager of this company desires to know as to how many units of each typewriter must be manufactured in the factory next month so as to maximize the profits of the company.

Formulate this as a linear programming problem and need not to be solved. **[C.A. (May) 93]**

Formulation. Let x_1, x_2 and x_3 be the number of *Manual, Electronic* and *Deluxe-Electronic* typewriters respectively which are to be manufactured in the factory next month. From the given data, profit contribution per unit in rupees will be (4,100–2500), (7,500–4,500) and (14,600–9,000) for Manual, Electronic and Deluxe–Elec. typewriters, respectively, *i.e.* Rs. 1600, Rs. 3000 and Rs. 5,600 respectively. Therefore, the objective function is given by:

$$\text{Maximize } P = 1600x_1 + 3{,}000x_2 + 5{,}600x_3.$$

From the data given for the time required for various models, we get the following constraints:

$$15x_1 + 12x_2 + 14x_3 \le 3{,}000 \text{ (machine time restriction)}$$

and $$4x_1 + 3x_2 + 5x_3 \le 1{,}200 \text{ (assembly time restriction)}$$

The level of operations in the company is subject to the availability of cash next month.

The cash requirements for x_1 units of Manual, x_2 units of Electronic and x_3 units of Deluxe-Electronic typewriters are:

$$2{,}500x_1 + 4{,}500x_2 + 9{,}000x_3 \qquad \text{...(1)}$$

The cash availability for the next month from the balance sheet is as follows:

Cash availability (Rs.) = Cash balance (Rs. 1,40,000) + Receivables (Rs. 50,000)
– Loan to repay to co-operative bank (Rs. 40,000)
– Interest on loan from TNC & Co-op banks (Rs. 1200)
– Interest on long term loans $\left(\dfrac{0{\cdot}24 \times \text{Rs. } 1{,}00{,}000}{12}\right)$
– Top management salary & fixed overheads Rs. 10,000

That is, Cash availability = Rs. 1,40,000 + Rs. 50,000 – Rs. (40,000 + 1200 + 2000 + 10,000)
= Rs. 1,90,000 – Rs. 53,200 = Rs. 1,36,800 ...(2)

From (1) and (2), we get the constraint

$$2{,}500\,x_1 + 4{,}500\,x_2 + 9{,}000\,x_3 \le 1{,}36{,}800.$$

Also, the company has promised to deliver 2 Manual and 8 Deluxe-Elec. typewriters to one on its customers. Hence $x_1 \ge 2, x_2 \ge 0,$ and $x_3 \ge 8$.

Finally, the formulated LPP can be put in the following form.

Max. $P = 1{,}600x_1 + 3{,}000x_2 + 5{,}600x_3$, subject to the constraints:

$$15x_1 + 12x_2 + 14x_3 \le 3000,\ 4x_1 + 3x_2 + 5x_3 \le 1{,}200,\ 2500x_1 + 4500x_2 + 9000x_3 \le 1{,}36{,}800$$

$x_1 \ge 2, x_2 \ge 0, x_3 \ge 8$ and x_1, x_2, x_3 can take positive integral values only.

Example 16. *The most recent audited summarized Balance Sheet of Shop and Shop Financial services is given below:*

Balance Sheet as on March 31, 1994

Liabilities	(Rs. in lakhs)	Assets	(Rs. in lakhs)
Equity Share Capital	65	**Fixed Assets:**	
Reserves & Surplus	110	Assets on Lease (original cost : Rs. 550 lakhs)	375
Term Loan from IFCI	80	Other Fixed Assets	50
Public Deposites	150	Investments (on wholly owned subsidiaries)	20
Bank Borrowings	147	**Current Assets:**	
Other Current Liabilities	50	Stock on Hire	80
		Receivables	30
		Other Current Assets	35
		Miscellaneous expenditure (not written off)	12
Total	602	Total	602

The company intends to enhance its investment in the lease portfolio by another Rs. 1,000 lakhs. For this purpose it would like to raise a mix of debt and equity in such a way that the overall cost of raising additional funds is minimized. The following constraints apply to the way the funds can be mobilized:

(1) Total debt divided by net owned funds, cannot exceed 10.

(2) Amount borrowed from financial institutions cannot exceed 25% of the net worth.

(3) Maximum amount of bank borrowings cannot exceed three times the net owned funds.

(4) The company would like to keep the total public deposit limited to 40% of the total debt.

The post-tax costs of the different sources of finance are as follows :

Equity	Term Loans	Public Deposits	Bank Borrowings
2·5%	8·5%	7%	10%

Formulate the funding problem as LPP.

Note. (a) Total Debt = Term Loans from Financial Institutions + Public Deposites + Bank Borrowings.

(b) Net Worth = Equity Share Capital + Reserves & Surplus.

(c) Net Owned Funds = Net Worth – Miscellaneous Expenditures. **[CA. (May, 94)]**

Formulation. Let x_1, x_2, x_3 and x_4 be the quantity of additional funds (in lakhs) raised on account of additional equity, additional term loans, additional public deposites and additional bank borrowings, respectively. The objective function is to minimize the cost of additional funds raised by the company. That is,

Minimize $C = 0·25x_1 + 0·85x_2 + 0·07x_3 + 0·1x_4$, subject to the following constraints :

(1) $\left[\dfrac{\text{Total Debt}}{\text{Net owned funds}}\right] \le 10$ or $\left[\dfrac{\text{(Existing debt + Additional)}}{\text{(Equity share capital + Reserve \& Surplus + Additional Equity – Misc. Exp.)}}\right] \le 10$

or $\dfrac{80+150+147+x_2+x_3+x_4}{(65+110+x_1)-12} \le 10$ or $\dfrac{x_2+x_3+x_4+377}{x_1+163} \le 10$

or $x_2+x_3+x_4+377 \le 10x_1+1630$ or $-10x_1+x_2+x_3+x_4 \le 1253.$

(2) Amount borrowed (financial institutions) ≤ 25% of net worth

or (Existing long term loan from financial institutions + Additional loan) ≤ 25% (Existing Equity Capital + Reserve & Surplus + Addl. Equity Capital)

or $80+x_2 \le 0·25\,(175+x_1)$ or $80+x_2 \le \frac{1}{4}(175+x_1)$

or $320+4x_2 \le 175+x_1$ or $-x_1+4x_2 \le -145$ or $x_1-4x_2 \ge 145$

(3) Maximum bank borrowings ≤ 3 (Net owned funds)

or (Existing bank borrowings + Addl. bank borrowings) ≤ 3 {(Existing Equity Capital + Reserves & Surplus + Addl: Equity Capital – Misc. Exp.}

or $(147+x_4) \le 3\,(65+110+x_1-12)$ or $x_4-3x_1 \le 525-36-147$ or $-3x_1+x_4 \le 342.$

(4) Total public deposit ≤ 40% of total debt

or (Existing public deposite + addl. public deposits) ≤ 0·40 (Existing total debt + Addl. total debt.)

or $150+x_3 \le 0·40\,(80+150+147+x_2+x_3+x_4)$ or $150+x_3 \le 0·40\,(x_2+x_3+x_4+377)$

or $1500+10x_3 \le 4x_2+4x_3+4x_4+1508$ or $-4x_2+6x_3-4x_4 \le 8.$

(5) Addl. equity capital + Addl. term loan + Addl. public deposits + Addl. bank borrowings = 1000 (since the company wants to enhance the investment by Rs. 1,000 lakhs)

or $$x_1+x_2+x_3+x_4 = 1000$$

(6) $x_1, x_2, x_3, x_4 \ge 0.$

Example 17. *Renco-Foundries is in the process of drawing up a Capital Budget for the next three years. It has funds to the time of Rs. 1,00,000 which can be allocated across the projects A, B, C, D and E. The net cash flows associated with an investment of Re. 1 in each project are provided in the following table.*

	Cash Flow at Time			
	0	1	2	3
From inv. A	– Re. 1	+ Re. 0·5	+ Rs. 1	Re. 0
From inv. B	Re. 0	– Re. 1	+ Re. 0·5	+ Re 1
From inv. C	– Re. 1	+ Rs. 1·2	Re. 0	Re. 0
From inv. D	– Re. 1	Re. 0	Re. 0	Rs. 1·9
From inv. E	Re. 0	Re. 0	– Re. 1	Rs. 1·5

Note : Time 0 = present, Time 1 = 1 year from now, Time 2 = 2 years from now, Time 3 = 3 years from now.

For example, Re. 1 invested in investment B requires a Re. 1 cash outflow at time 1 and returns Re. 0·50 at time 2 and Re. 1 at time 3.

To ensure that the firm remains reasonably diversified, the firm will not commit an investment exceeding Rs. 75,000 in any project. The firm cannot borrow funds; therefore the cash available for investment at any time is limited to cash on hand. The firm will earn interest at 8% per annum by parking the uninvested funds in *money market investments. Assume that the returns from investments can be immediately re-invested. For example, the positive cash flow received from project C at time 1 can immediately be reinvested in project B.*

Required : *Formulate an L.P. that will "Maximize cash on hand at time 3".* **[C.A. (Nov. 95)]**

Formulation. The company wants to decide optimum allocation of funds to project A, B, C, D, E and money market investments.

Let x_1, x_2, x_3, x_4 and x_5 be the amount of rupees invested in investments A, B, C, D and E respectively and s_i be the amount of Rs. invested in Money market investments at time i (for $i = 0, 1, 2$).

The objective of the company is to draw up the capital budget in such a way that will "maximize cash on hand at time 3", At time 3, the cash on hand for company will be the sum of all cash inflows at time 3.

Since the firm earns interest at 8% per annum by parking the uninvested funds in money market investments, hence Rs. s_0, Rs. s_1 and Rs. s_2 which are invested in these investments at times 0, 1 and 2 will become $1{\cdot}08\,s_0$, $1{\cdot}08s_1$ and $1{\cdot}08s_2$ at times 1, 2 and 3, respectively.

From the given table, it can be computed that at time 3 :

Cash on hand $= x_1 \times \text{Rs. } 0 + x_2 \times \text{Re } 1 + x_3 \times \text{Re } 0 + x_4 \times \text{Rs. } 1{\cdot}9 + x_5 \times \text{Rs. } 1{\cdot}5 + 1{\cdot}08s_2$

$= \text{Rs. } (x_2 + 1{\cdot}9x_4 + 1{\cdot}5x_5 + 1{\cdot}08\,s_2)$

The objective of the company is to maximize the cash at time 3. Hence the objective function will be :

$$\text{Max. } z = x_2 + 1{\cdot}9x_4 + 1{\cdot}5x_5 + 1{\cdot}08\,s_2 \quad \text{...(1)}$$

It may be remembered that :

$$\text{Cash available for investment at time } t = \text{cash on hand at time } t \quad \text{...(2)}$$

At time 0, funds to the tune of Rs. 1,00,000 are available for investment. From the table, it can be observed that funds are invested in investments A, C and D at time 0. Therefore,

$$x_1 + x_3 + x_4 + s_0 = 1{,}00{,}000 \quad \text{...(3)}$$

At time 1, Rs. $0{\cdot}5x_1$, Rs. $1{\cdot}2x_3$ and Rs. $1{\cdot}08\,s_0$ will be available as a result of investments made at time 0. From the table Rs. x_2 and Rs. s_1 are invested in investment B and money market investments, respectively at time 1.

Using (2), we get $$0{\cdot}5x_1 + 1{\cdot}2x_3 + 1{\cdot}08s_0 = x_2 + s_1 \quad \text{...(4)}$$

At time 2, Re $1\,x_1$, Rs. $0{\cdot}5x_2$ and Rs. $1{\cdot}08\,s_1$ will be available for investment. However, Rs. x_5 and Rs. s_2 are invested at time 2.

Hence, $$1x_1 + 0{\cdot}5x_2 + 1{\cdot}08\,s_1 = x_5 + s_2 \quad \text{...(5)}$$

Also, since the company will not commit an investment exceeding Rs. 75,000 in any project, we get the following constraints :

$$x_i \le 75{,}000 \text{ for } i = 1, 2, 3, 4, 5. \quad \text{...(6)}$$

and $x_1, x_2, x_3, x_4, x_5, s_0, s_1, s_2$ are all ≥ 0. Finally, combining all above constraints, the L.P. model for the Renco Foundries is obtained as given below.

Max. $z = x_2 + 1{\cdot}9x_4 + 1{\cdot}5x_5 + 1{\cdot}08s_2$, subject to the constraints :

$$x_1 + x_3 + x_4 + s_0 = 1{,}00{,}000,$$
$$0{\cdot}5x_1 + 1{\cdot}2x_3 + 1{\cdot}08s_0 = x_2 + s_1$$
$$1x_1 + 0{\cdot}5x_2 + 1{\cdot}08s_1 = x_5 + s_2$$
$$x_i \le 75{,}000 \ (i = 1, 2, 3, 4, 5)$$

and $x_1, x_2, x_3, x_4, x_5, s_0, s_1, s_2$ are all ≥ 0.

Example 18. *An agriculturist has a farm with 126 acres. He produces Radish, Muttar and Potato. Whatever he raises is fully sold in the market. He gets Rs. 5 for Radish per kg., Rs. 4 for Muttar per kg. and Rs. 5 for Potato per kg. The average yield is 1,500 kg. of Radish per acre, 1,800 kg. of Matter per acre and 1,200 kg. of Potato per acre. To produce each 100 kg. of Radish and Muttar and to produce each 80 kg. of Potato, a sum of Rs. 12·50 has to be used for manure. Labour required for each acre to raise the crop is 6 man-days for Radish and Potato each and 5 man-days for Muttar. A total of 500 man-days of labour at a rate of Rs. 40 per man-day are available. Formulate this as a linear programming model to maximize the agriculturist's total profit.* **[C.A. May 97]**

Solution. Let x_1, x_2, x_3 be the number of acres allotted for cultivating *Radish, Matter* and *Potato* respectively. Then the following table can be worked out.

	Radish	Muttar	Potato
Selling Price	5 × 1,500 = 7,500	4 × 1,800 = 7,200	5 × 1,200 = 6,000
Manure cost	12·50 × 15 = 187·50	12·50 × 18 = 225	1250 × 15 = 187·50
Labour cost	40 × 6 = 240	40 × 5 = 200	40 × 6 = 240
Profit	7,500 − 427·50 = 7,072·50	7,200 − 425 = 6,775	6,000 − 427·50 = 5,572·50

The linear programming formulation of the given problem is as follows :

Max. (Total profit) $P = 7072{\cdot}5\,x_1 + 6775\,x_2 + 5572{\cdot}5\,x_3$, subject to the constraints

$$x_1 + x_2 + x_3 \le 125 \qquad \textit{(land constraint)}$$
$$6x_1 + 5x_2 + 6x_3 \le 500 \qquad \textit{(labour constraint)}$$
$$x_1, x_2, x_3 \ge 0.$$

Example 19. *A company has three operational departments (weaving, processing and packing) with capacity to produce three different types of cloths namely suitings, shirtings, and woolens yielding the profit of Rs. 2, Rs. 4 and Rs. 3 per meter respectively. One meter suiting requires 3 minutes in weaving, 2 minutes in processing and 1 minute in packing 1 meter of shirting requires 4 minutes in weaving, 1 minute in processing and 3 minutes in packing while one meter woolen requires 3 minutes in each department. In a week, total run time of each department is 60, 40 and 80 hours of weaving, processing and packing departments respectively.*

Formulate the linear programming problem to find the product mix to maximize the profit. **[C.A. Nov. 98]**

Solution. The problem may be expressed as follows :

Reserouces / Constraints	Product			Total availability (minutes)
	Suiting	Shirting	Woolen	
Weaving deptt.	3	4	3	60 × 60
Processing deptt.	2	1	3	40 × 60
Packing deptt.	1	3	3	80 × 60

The linear programming formulation can be easily obtained as follows :

Max. (Total profit) $P = 2x_1 + 4x_2 + 3x_3$,

Subject to the constraints :

$$3x_1 + 4x_2 + 3x_3 \le 3{,}600$$
$$2x_1 + x_2 + 3x_3 \le 2{,}400$$
$$x_1 + 3x_2 + 3x_3 \le 4{,}800$$

and $x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0.$

Example 20. *A firm buys castings of* P and Q *type of parts and sells them as finished product after machining, boring and polishing. The purchasing cost for castings are Rs. 3 and Rs. 4 each for parts P* and Q *and selling costs are Rs. 8 and Rs. 10 respectively. The per hour capacity of machines used for machining, boring and polishing for two products is given below :*

Capacity per hour	Parts	
	P	Q
Machining	30	50
Boring	30	45
Polishing	45	30

The running costs for machining, boring and polishing are Rs. 30, Rs. 22·5 and Rs. 22·5 per hour respectively. FORMULATE the linear programming problem to find out the product mix to maximize the profit. **[C.A. Nov. 97]**

Solution. Let x_1, x_2 be the number of P and Q type parts to be produced per hour respectively.

For the profit part of x_1 and x_2, we calculate the total cost for each and then subtract the sale price of that part from it. The cost and profit per part are calculated in the following table :

Casting operation	x_1	x_2
Machining	30/30 = 1·00	30/50 = 0·60
Boring	22·5/30 = 0·75	22·5/45 = 0·50
Polishing	22·5/45 = 0·50	22·5/30 = 0·75
Purchase	3	4
Total cost	5·25	5·85
Sale price	8	10
Profit	2·75	4·15

On the machine, type of P parts consumes 1/30th of the available hour, a type Q part consumes 1/50th of an hour. Thus, the machine constraint becomes :

$$\frac{1}{30}x_1 + \frac{1}{50}x_2 \le 1 \quad \text{or} \quad 50x_1 + 30x_2 \le 1{,}500$$

Similarly, other constraints can be established.

The linear programming formulation of the given problem will be :

Max. (Total profit) $P = 2{\cdot}75x_1 + 4{\cdot}15x_2$,

Subject to the constraints :

$\frac{1}{30}x_1 + \frac{1}{50}x_2 \le 1$ or $50x_1 + 30x_2 \le 1{,}500$ **(machine constraint)**

$\frac{1}{30}x_1 + \frac{1}{45}x_2 \le 1$ or $45x_1 + 30x_2 \le 1{,}350$ **(boring constraint)**

$\frac{1}{45}x_1 + \frac{1}{30}x_2 \le 1$ or $30x_1 + 45x_2 \le 1{,}350$ **(polishing constraint)**

and $x_1, x_2 \ge 0$.

Example 21. *The owner of Fancy Goods shop is interested to determine how many advertisements to release in the selected three magazines, A, B* and *C. His main purpose is to advertise in such a way that total exposure to principal buyers of his goods is maximised. Percentages of readers for each magazine are known. Exposure in any particular magazine is the number of advertisements released multiplied by the number of principal buyers. The following data are available :*

Particulars	*Magazines*		
	A	B	C
Readers	1·0 lakh	0·6 lakh	0·4 lakh
Principal buyers	20%	15%	8%
Cost per Adv.	8,000	6,000	5,000

The budgeted amount is at the most Rs. 1·0 *lakh for the advertisements. The owner has already decided that magazine A should have no more than 15 advertisements and that B* and *C each gets at least 8 advertisements. FORMULATE a linear programming model for this problem. DO NOT SOLVE.* **[C.A. Nov. 96]**

Solution. Let x_1, x_2 and x_3 be the required number of insertions in magazine A, B and C respectively. The total exposure of principal buyers of the magazine is

$$Z = (20\% \text{ of } 1{,}00{,}000)\,x_1 + (15\% \text{ of } 60{,}000)\,x_2 + (8\% \text{ of } 40{,}000)\,x_3$$

$$8{,}000\,x_1 + 6{,}000\,x_2 + 5{,}000\,x_3 \le 1{,}00{,}000 \qquad \textit{(budgeting constraint)}$$
$$x_1 \le 15,\; x_2 \ge 8 \text{ and } x_3 \ge 8 \qquad \textit{(advertisement constraint)}$$

The LP model is :

Max. (Total exposure) $P = 20{,}000\,x_1 + 9{,}000\,x_2 + 3{,}200\,x_3$,

subject to the constraints :

$$8{,}000\,x_1 + 6{,}000\,x_2 + 5{,}000\,x_3 \le 1{,}00{,}000$$
$$0 \le x_1 \le 15,\; 0 \le x_2 \le 8,\; 0 \ge x_3 \ge 8.$$

Example 22. *Evening shift resident doctors in a Govt. hospital work five consecutive days and have two consecutive days off. Their five days of work can start on any day of the week and the schedule rotates indefinitely. The hospital requires the following minimum number of doctors working :*

Sun.	Mon.	Tues.	Wed.	Thurs.	Fri.	Sat.
35	55	60	50	60	50	45

No more than 40 doctors can start their five working days on the same day. Formulate the general linear programming model to minimize the number of doctors employed by the hospital. **[Delhi (MBA) 98]**

Solution. Let $x_1, x_2, x_3, x_4, x_5, x_6, x_7$ be the number of doctors who start their duty on j^{th} ($j = 1, 2, \ldots, 7$) day of the week. The given problem has the following LP formulation :

Max. (Total no. of doctors) $P = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7$,

subject to the constraints :

$$x_1 + x_4 + x_5 + x_6 + x_7 \ge 35$$
$$x_2 + x_5 + x_6 + x_7 + x_1 \ge 55$$
$$x_3 + x_6 + x_7 + x_1 + x_2 \ge 60$$
$$x_4 + x_7 + x_1 + x_2 + x_3 \ge 50$$
$$x_5 + x_1 + x_2 + x_3 + x_4 \ge 60$$
$$x_6 + x_2 + x_3 + x_4 + x_5 \ge 50$$
$$x_7 + x_3 + x_4 + x_5 + x_6 \ge 45$$
$$0 \le x_j \le 40\; (j = 1, 2, \ldots, 7).$$

Example 23. *The Omega Data Processing Company performs three types of activity : pay rolls, account receivables, and inventories. The profit and time requirements for key punch computation and office printing for a standard job are shown in the following table :*

Omega guarantees over night completion of the job. Any job schedule during the day can be completed during the day or night. Any job scheduled during the night, however, must be completed during the night. The capacity for both day and night are shown in the following table :

Capacity (Min.)	*Key punch*	*Computation*	*Print*
Day	4,200	150	400
Night	9,200	250	650

Formulate the linear programming problem in order to determine the 'mixture' of standard jobs that should be accepted during the day and night.

Solution. Let x_{ij} represent the jobs accepted during day and night.

The LP model is :

$$\text{Max. } P = 275\,(x_{11} + x_{12}) + 125\,(x_{21} + x_{22}) + 225\,(x_{31} + x_{32}),$$

subject to the constraints :

$$1200\,(x_{11} + x_{12}) + 1400\,(x_{21} + x_{22}) + 800\,(x_{31} + x_{32}) \ge 13{,}400$$
$$100\,(x_{11} + x_{12}) + 60\,(x_{21} + x_{22}) + 80\,(x_{31} + x_{32}) \le 1{,}050$$
$$1{,}200\,x_{12} + 1{,}400\,x_{22} + 800\,x_{32} \le 9{,}200$$
$$100\,x_{12} + 60x_{22} + 80\,x_{32} \le 650$$

and $$x_{ij} \ge 0 \text{ for all } i \text{ and } j.$$

Example 24. *PQR Coffee Company mixes South Indian, Assamese and Imported coffee to make two brands of coffee, Plains X and Plains XX. The characteristics used in blending the coffees include strength, acidity and caffeine. The test results of the available supplies of South Indian, Assamese, and Imported coffees are shown in the following table :*

	Price/kg (Rs.)	Strength index	Acidity index	Percent caffeine	Supply available (kgs.)
South Indian	30	6	4·0	2·0	40,000
Assamese	40	8	3·0	2·5	20,000
Imported	35	5	3·5	1·5	15,000

The requirements for Plains X and Plains XX coffees are given in the following table :

Plains Coffee	Price/kg (Rs.)	Min. Strength	Max. Acidity	Max. per cent caffeine	Quantity Demanded (kgs)
X	45	6·5	3·8	2·2	35,000
XX	55	6·0	3·5	2·0	25,000

Assume that 35,000 kgs of Plains X and 25,000 kgs. of Plains XX are profits to be sold. Formulate the LPP to maximize profit. **[Bombay (MMS) 96]**

Solution. For Plain Coffee *X*, let

x_{11} = quantity in kg of South Indian coffee

x_{12} = quantity in kg of Assamese coffee

x_{13} = quantity in kg of Imported coffee

and for Plain Coffee *XX*

x_{21} = quantity in kg of South Indian coffee

x_{22} = quantity in kg of Assamese coffee

x_{23} = quantity in kg of Imported coffee

Thus, the LP model can be presented as follows :

$$\text{Max. (Profit) } P = 45\,(30x_{11} + 40x_{12} + 35x_{13}) + 55\,(30\,x_{21} + 40\,x_{22} + 35\,x_{23}),$$

subject to the constraints

$$\left.\begin{array}{l} 6x_{11} + 8x_{12} + 5x_{13} \ge 6{\cdot}5 \\ 6x_{21} + 8x_{22} + 5x_{23} \ge 6{\cdot}0 \end{array}\right\} \textit{strength index constraint}$$

$$\left.\begin{array}{l} 4x_{11} + 3x_{12} + 3{\cdot}5\,x_{13} \le 3{\cdot}8 \\ 4x_{21} + 3x_{22} + 3{\cdot}5\,x_{23} \le 3{\cdot}5 \end{array}\right\} \textit{acidity constraint}$$

$$\left.\begin{array}{l} 2x_{11} + 2{\cdot}5\,x_{12} + 1{\cdot}5\,x_{13} \le 2{\cdot}2 \\ 2x_{21} + 2{\cdot}5\,x_{22} + 2{\cdot}5\,x_{23} \le 2{\cdot}0 \end{array}\right\} \textit{caffeine constraint}$$

$$\left.\begin{array}{l} x_{11} + x_{21} \le 40{,}000 \\ x_{12} + x_{22} \le 20{,}000 \\ x_{13} + x_{23} \le 15{,}000 \end{array}\right\} \textit{sale constraint}$$

$$\left.\begin{array}{l} x_{11} + x_{12} + x_{13} = 35{,}000 \\ x_{21} + x_{22} + x_{23} = 25{,}000 \end{array}\right\} \textit{sale constraint}$$

$$x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23} \ge 0.$$

Example 25. *A manufacturer of biscuits is considering four types of gift packs containing three types of biscuits : orange cream (OC), chocolate cream (CC), and Wafors (W). Market research study conducted recently to assess the preferences of the consumers shows the following types of assortments to be in good demand :*

Assortments	*Contents*	*Selling price per kg.*
A	Not less than 40% of OC Not more than 20% of CC Any quantity of W	20
B	Not less than 20% of OC Not more than 40% of CC Any quantity of W	25
C	Not less than 50% of OC Not more than 10% of CC Any quantity of W	22
D	No restrictions	12

For the biscuits, the manufacturing capacity and costs are given below :

Biscuits variety	*Plant capacity* (kg/day)	*Manufacturing cost* Rs/kg.
OC	200	8
CC	200	9
W	150	7

Formulate a linear programming model to find the production schedule which maximizes the profit assuming that there are no market restrictions. **[Delhi (MBA) April, 99]**

(Note. This is also discussed earlier in Ex. 11 on page 61.)

Solution. Let the decision variables x_{ij} $(i = A, B, C, D; j = 1, 2, 3)$ be defined as follows :

(i) x_{A1}, x_{A2}, x_{A3}, denote the quantity in kg of OC, CC and W type of biscuits respectively for A.

(ii) x_{B1}, x_{B2}, x_{B3}, denote the quantity in kg of OC, OC and W type of biscuits, respectively, for B.

(iii) x_{C1}, x_{C2}, x_{C3}, denote the quantity in kg. of OC, CC and W type of biscuits respectively, for C.

(iv) x_{D1}, x_{D2}, x_{D3} denote the quantity in kg of OC, CC and W type of biscuits, respectively, for D.

The LP model can be easily obtained as :

$$\text{Max. } Z = 20\,(x_{A1} + x_{A2} + x_{A3}) + 25\,(x_{B1} + x_{B2} + x_{B3}) + 22\,(x_{C1} + x_{C2} + x_{C3})$$
$$+\,12\,(x_{D1} + x_{D2} + x_{D3}) - 8\,(x_{A1} + x_{B1} + x_{C1} + x_{D1}) - 9\,(x_{A2} + x_{B2} + x_{C2} + x_{D2}) - 7\,(x_{A3} + x_{B3} + x_{C3} + x_{D3})$$
$$= 12x_{A1} + 11x_{A2} + 13x_{A3} + 17x_{B1} + 16x_{B2} + 18x_{B3} + 24x_{C1} + 13x_{C2} + 15x_{C3} + 4x_{D1} + 3x_{D2} + 5x_{D3}$$

subject to the constraints :

$$\left.\begin{aligned} x_{A1} &\geq 0{\cdot}40\,(x_{A1} + x_{A2} + x_{A3}) \\ x_{A2} &\leq 0{\cdot}20\,(x_{A1} + x_{A2} + x_{A3}) \end{aligned}\right\} \text{(constraints for } A)$$
$$\left.\begin{aligned} x_{B1} &\geq 0{\cdot}20\,(x_{B1} + x_{B2} + x_{B3}) \\ x_{B2} &\leq 0{\cdot}40\,(x_{B1} + x_{B2} + x_{B3}) \end{aligned}\right\} \text{(constraints for } B)$$
$$\left.\begin{aligned} x_{C1} &\geq 0{\cdot}50\,(x_{C1} + x_{C2} + x_{C3}) \\ x_{C2} &\leq 0{\cdot}20\,(x_{C1} + x_{C2} + x_{C3}) \end{aligned}\right\} \text{(constraints for } C)$$
$$\left.\begin{aligned} x_{A1} + x_{B1} + x_{C1} + x_{D1} &\leq 200 \\ x_{A2} + x_{B2} + x_{C2} + x_{D2} &\leq 200 \\ x_{A3} + x_{B3} + x_{C3} + x_{D3} &\leq 150 \end{aligned}\right\} \text{(plant capacity constraints)}$$
$$x_{ij} \geq 0 \quad (i = A, B, C, D;\ j = 1, 2, 3)$$

EXAMINATION PROBLEMS

1. A firm manufactures headache pills in two sizes A and B. Size A contains 2 grains of aspirin, 5 grains of bicarbonate and 1 grain of codeine. Size B contains 1 grain of aspirin, 8 grains of bicarbonate and 6 grains of codeine. It is found by users that it requires at least 12 grains of aspirin, 74 grains of bicarbonate and 24 grains of codeine for providing immediate effect. It is required to determine the least number of pills a patient should take to get immediate relief. Formulate the problem as a standard LPP. **[Kota 91]**
   [Ans. Min. $z = x_1 + x_2$, *s.t.* $2x_1 + x_2 \geq 12$, $5x_1 + 8x_2 \geq 74$, $x_1 + 6x_2 \geq 24$; $x_1, x_2 \geq 0$.]
2. A manufacturer has three machines A, B, C with which he produces three different articles P, Q, R. The different machine times required per article, the amount of time available in any week on each machine and the estimated profits per article are furnished in the following table :

Article	*Machine time (in hrs.)*			*Profit per article (in rupees)*
	A	B	C	
P	8	4	2	20
Q	2	3	0	6
R	3	0	1	8
Available machine hrs.	250	150	50	

Formulate the problem as a linear programming problem.

[**Ans.** Max. $z = 20P + 6Q + 8R$; such that $8P + 2Q + 3R \le 250$, $4P + 3Q \le 150$, $2P + R \le 50$; and $P, Q, R \ge 0$.]

3. Two alloys A and B are made from four different metals I, II, III and IV according to the following specifications :

A : at most 80% of I, at most 30% of II, at least 50% of III, B : between 40% & 60% of II, at least 30% of III, at most 70% of IV.

The four metals are extracted from three different ores whose consituents percentage of these metals, maximum available quantity and cost per ton are as follows :

Constituent %

Ore	*Max. Quantity (tons)*	I	II	III	IV	Others	Price (Rs. per ton)
1	1000	20	10	30	30	10	30
2	2000	10	20	30	30	10	40
3	3000	5	5	70	20	0	50

Assuming the selling prices of alloys A and B are Rs. 200 and Rs. 300 per ton respectively. Formulate the above as a linear programming problem selecting appopriate objective and constraint functions. [I.C.W.A. (Dec.) 88]

[**Ans.** Max. $z = 200A + 300B - 30\,(x_{1A} + x_{2B}) - 40\,(x_{2A} + x_{2B}) - 50\,(x_{3A} + x_{3B})$, subject to the constraints :

$$\left.\begin{array}{l} 0{\cdot}20x_{1A} + 0{\cdot}10x_{2A} + 0{\cdot}05x_{3A} \le 0{\cdot}80A \\ 0{\cdot}10x_{1A} + 0{\cdot}20x_{2A} + 0{\cdot}05x_{3A} \le 0{\cdot}30A \\ 0{\cdot}30x_{1A} + 0{\cdot}30x_{2A} + 0{\cdot}70x_{3A} \ge 0{\cdot}50A \end{array}\right] \text{(alloy specification for } A)$$

$$\left.\begin{array}{l} 0{\cdot}1x_{1B} + 0{\cdot}2x_{2B} + 0{\cdot}05x_{3B} \ge 0{\cdot}40B \\ 0{\cdot}1x_{1B} + 0{\cdot}2x_{2B} + 0{\cdot}05x_{3B} \le 0{\cdot}6B \\ 0{\cdot}3x_{1B} + 0{\cdot}3x_{2B} + 0{\cdot}7x_{3B} \ge 0{\cdot}3B \\ 0{\cdot}3x_{1B} + 0{\cdot}3x_{2B} + 0{\cdot}2x_{3B} \le 0{\cdot}7B \end{array}\right] \text{(alloy specification for } B)$$

$$\left.\begin{array}{l} 0{\cdot}6x_{1A} + 0{\cdot}6x_{2A} + 0{\cdot}3x_{3A} = A \\ 0{\cdot}7x_{1B} + 0{\cdot}8x_{2B} + 0{\cdot}95x_{3B} = B \end{array}\right\} \text{(material balance)}$$

$$\left.\begin{array}{l} x_{1A} + x_{1B} \le 1000 \\ x_{2A} + x_{2B} \le 2000 \\ x_{3A} + x_{3B} \le 3000 \end{array}\right\} \text{(availability of ores)}$$

$A, B, x_{iA}, x_{iB} \ge 0$ $(i = 1, 2, 3)$, where A and B denote quantities of alloy of A and B; and x_{iA}, x_{iB} $(i = 1, 2, 3)$ denote the quantities going into A or B from I, II, III, or IV.]

4. Consider the following problem faced by a production planner in a soft drink plant. He has two bottling machines A and B. A is designed for 8-ounce bottles and B for 16-ounce bottles. However, each can be used on both types with some loss of efficiency. The following data is available.

Machine	*8-ounce bottles*	*16-ounce bottles*
A	100/minute	40/minute
B	60/minute	75/minute

The machine can be run 8-hour per day, 5 days per week. profit on 8-ounce bottle is 15 paise and on 16-ounce is 25 paise. Weekly production of the drink cannot exceed 3,00,000 ounces and the market can absorb 25,000 eight-ounce bottles and 7,000 sixteen-ounce bottles per week. The planner wishes to maximize his profit subject, of course, to all the production and marketing restrictions, formulate this as linear programming problem.

[**Hint.** Let x_1 units of 8-ounce bottle and x_2 units of 16-ounce bottle be produced. Total profit of production planner is given by $P = 0.15x_1 + 0.25x_2$.

Since machine A and B both work 8 hours per day and 5 days per week, the total working time for machine A and B will become 2400 minutes per week. Therefore, time conditions will be

$$\frac{x_1}{100} + \frac{x_2}{40} \le 2400 \text{ (for machine } A), \quad \frac{x_1}{60} + \frac{x_2}{75} \le 400 \text{ (for machine } B)$$

Restriction of total weekly production will be $8x_1 + 16x_2 \le 3{,}00{,}000$.

Market consumption is restricted by $0 \le x_1 \le 25000$ and $0 \le x_2 \le 7000$.

5. Formulate the following linear programming problem.

A used-car dealer wishes to stock-up his lot to maximize his profit. He can select cars A, B and C which are valued wholesale at Rs. 5000, Rs. 7000 and Rs. 8000 respectively. These can be sold at Rs. 6000, 8500 and 10500 respectively.

For each car type, the probabilities of sale are :

Type of car	:	A	B	C
Prob. of sale in 90 days.	:	0·7	0·8	0·6

For every two cars of B, he should buy one car of type A or C. If he has Rs, 1,00,000 to invest, what should he buy to maximize his expected gain. [VTU (BE VIth Sem.) Aug. 2002]

[Hint. Let x_1, x_2, x_3 number of cars be purchased of type A, B, C respectively. Gain per car for A, B, C will be Rs. (6000 – 50000), Rs. (85000 – 7000), Rs. (10500 – 8000) respectively. Therefore total expected gain will be

$$z = 1000\, x_1 \times 0{\cdot}7 + 1500 x_2\, 0{\cdot}8 + 2500 x_3 \times 0{\cdot}6]$$

Investment constraints will be given by

$$5000 x_1 + 7000\,(2x_2) \le 1{,}00{,}000 \text{ and } 7000\,(2x_2) + 8000\, x_3 \le 1{,}00{,}000].$$

6. Certain farming organization operated three farms of comparable productivity. The output of each farm is limited both by the usable acreage and by the amount of water available for irrigation. The data for the upcoming season are the following :

Farm	*Usable acreage*	*Water available in acre feet*
1	400	1500
2	600	2000
3	300	900

The organization is considering three crops for planting which differ primarily in their expected profit per acre and in their consumption of water. Furthermore, the total acreage that can be devoted to each of the crops is limited by the amount of appropriate harvesting equipment available.

Crop	**Minimum acreage**	**Water consumption in acre feet**	**Expected profit per acre**
A	700	5	Rs. 400
B	800	4	Rs. 300
C	300	3	Rs. 100

In order to maintain a uniform work-load among the farms, it is the policy of the organization that the percentage of the usable acreage planted must be the same at each farm. However, any combination of the crops may be grown at any of the farms. The organization wishes to know how much of each crop should be planted at the respective farms in order to maximize expected profit. Formulate this as a linear programming problem. **[Roorkee (B.E. IV th) 91; Delhi (M.B.A.) 75]**

[Hint. Let the number of acres at the ith farm devoted to jth crop be denoted by the decision variable x_{ij} ($i = 1, 2, 3$; $j = A, B, C$).

Then the formulation of the problem will be :

$$\text{Max. } P = 400 \sum_{i=1}^{3} x_{iA} + 300 \sum_{i=1}^{3} x_{iB} + 100 \sum_{i=1}^{3} x_{iC}, \quad \text{subject to, } \sum_{j=A}^{C} x_{1j} \le 400,\ \sum_{j=A}^{C} x_{2j} \le 600,\ \sum_{j=A}^{C} x_{3j} \le 300.$$

$$\left\{\begin{array}{l} 5x_{1A} + 4x_{1B} + 3x_{1C} \le 1{,}5000 \\ 5x_{2A} + 4x_{2B} + 3x_{2C} \le 2{,}000 \\ 5x_{3A} + 4x_{3B} + 3x_{3C} \le 900 \end{array}\right\}, \quad \sum_{j=A}^{C} x_{1j} \le 700,\ \sum_{j=A}^{C} x_{2j} \le 800,\ \sum_{j=A}^{C} x_{3j} \le 300,$$

As the farming organization wishes to keep the policy of uniform workload, the following equations must also hold :

$$\frac{x_{1A} + x_{1B} + x_{1C}}{400} = \frac{x_{2A} + x_{2B} + x_{2C}}{600},\ \frac{x_{2A} + x_{2B} + x_{2C}}{600} = \frac{x_{3A} + x_{3B} + x_{3C}}{300},\ \frac{x_{1A} + x_{1B} + x_{1C}}{400} = \frac{x_{3A} + x_{3B} + x_{3C}}{300}.$$

Since the first two equations yield the third, the third equation can be omitted from the model. Now rearranging the remaining two equations, the uniform workload restrctions become :

$$3\,(x_{1A} + x_{1B} + x_{1C}) - 2\,(x_{2A} + x_{2B} + x_{2C}) = 0 \text{ and } (x_{2A} + x_{2B} + x_{2C}) - 2\,(x_{3A} + x_{3B} + x_{3C}) = 0.$$

7. A feed mixing company purchases and mixes one or mere of the three types of grain, each containing different amount of three nutritional elements, the data is given below :

Item Nutritional ingredient	*One unit weight of*			*Minimum total requirement, over planning horizon*
	Grain I	*Grain 2*	*Grain 3*	
A	2	4	6	≥ 125
B	0	2	5	≥ 24
C	5	1	3	≥ 80
Cost per unit wt. (Rs.)	25	15	18	Minimize

The production manager specifies that any feed mix for his livestock must meet at least minimal nutritional requirements; and seeks the least costly among all such mixes. Suppose his planning horizon is a two week period, that is, he purchases enough to fill his needs for two weeks. Formulate this as an L.P.P.

[Hint. Let x_1, x_2, x_3 denote the weight levels of three different grains. Then by considering the nutritional ingredient in the three grains, linear programming problem is :

To minimize $C = 25x_1 + 15x_2 + 18x_3$, subject to the constraints :

$$2x_1 + 4x_2 + 6x_3 \ge 125,\ 2x_2 + 5x_3 \ge 24,\ 5x_1 + x_2 + 3x_3 \ge 80, \text{ and } x_1, x_2, x_3 \ge 0]$$

8. *ABC* foods company is developing a low calorie high protein diet supplement called *Hi*-Pro. The specification for *Hi*-Pro, have been established by a panel of medical experts. These specifications along with the calorie, protein and vitamin content of three basic foods are given in the following table :

Units of Nutritional Elements per 100 gm Serving of Basic Foods :

Nutritional Elements	Basic Foods			Hi-Pro. Specifications
	No. 1	No. 2	No. 3	
Calories	350	250	200	≤ 300
Protein	250	300	150	≥ 200
Vitamin A	100	150	75	≥ 100
Vitamin C	75	125	150	≥ 100
Cost per serving (Rs.)	1.50	2·00	1·20	

Formulate the linear programming model to minimize cost.

[Hint. Let x_1 ; x_2, x_3 denote the number of units of basic food number 1, 2 and 3, respectively. Then the formulation will be obtained as follows : Min. $z = 1{\cdot}50x_1 + 2x_2 + 1{\cdot}20x_3$, subject to the conditions :

$350x_1 + 250x_2 + 200x_3 \le 300$ $\quad$ $100x_1 + 150x_2 + 75x_3 \ge 100$ $\quad$ $x_1, x_2, x_3 \ge 0$.]

$250x_1 + 300x_2 + 150x_3 \ge 200$ $\quad$ $75x_1 + 125x_2 + 150x_{23} \ge 100$

9. **(Investment Problem)** Mr. Krishnamutry, a retired Govt. officer, has recently received his retirement benefits, *viz.*, provident fund, gratuity, etc. He is contemplating as to how much funds he should invest in various alternatives open to him so as to maximize return on investment. The investment alternatives are : government securities, fixed deposits of a public limited company, equity shares, time deposits in banks, national saving certificates and real estate. He has made a subjective estimate of the risk involved on a five point scale. The data on the return on investment, the number of years for which the funds will be blocked to earn this return on investment and the subjective risk involved are as follows :

Investment Alternates	Return	Number of years	Risk
Govt. securities	6%	15	1
Company deoposits	15%	3	3
Equity shares	20%	6	7
Time deposits	10%	3	1
NSC	12%	6	1
Real Estate	25%	10	2

He was wondering what percentage of funds he should invest in each alternative so as to maximize the return on investment. He decided that average risk should not be more than 4, and funds should not be locked up for more than 15 years. He would necessarily invest at least 30% in real estate.

Formulate an LP model for the problem.

[Hint Let x_1 , x_2, x_3, x_4 and x_5 be the percentage of the total funds that should be invested in government securities, company deposits, equity shares, time deposits, national saving certificates and real estate, respectively. The LP model can be formulated as follows :

Maximize $z = 0{\cdot}06x_1 + 0{\cdot}15x_2 + 0{\cdot}20x_3 + 0{\cdot}10x_4 + + 0{\cdot}12x_5 + 0{\cdot}25x_6$, subject to the conditions :

$15x_1 + 3x_2 + 6x_3 + 3x_4 + 6x_5 + 10x_6 \le 15,\ x_1 + 3x_2 + 7x_3 + x_4 + x_5 + 2x_6 \le 4,\ x_1, x_2, x_3 \ge 0$.

$x_6 \ge 0{\cdot}30 ;\ x_1, x_2, x_3, x_4, x_5, x_6 \ge 0$].

10. **(Media Selection Problem)** The Owner of Metro Sports wishes to determine how many advertisements to place in the selected three monthly magazines A, B and C. His objective is to advertise in such a way that total exposure to principal buyers of expensive sports goods is maximized. Percentages of readers for each magazine are known, Exposure in any particular magazine is the number of advertisments placed multiplied by the number of principal buyers. The following data may be used :

Exposure Category	Magazines		
	A	B	C
Readers (in Lakhs)	1	0·6	0·4
Principal Buyers	10%	15%	7%
Cost per advertisement (Rs.)	5000	4500	4250

The budgeted amount is at most Rs. 1 Lakh for the advertisements. The owner has already decided that magazine A should have no more than 6 advertisements and that B and C each have at least two advertisements. Formulate a LP model for the problem.

[Hint. Let x_1, x_2, x_3 be the number of insertions in magazine A, B and C, respectively.

Then the LP formulation will be as follows :

Maximize : z = (10% of 1,00,000) x_1 + (15% of 60,000) x_2 + (7% of 40,000) x_3 (Total exposure)

subject to the constraints : $5000\ x_1 + 4500\ x_2 + 4250\ x_3 \le 1{,}00{,}000$ (Budgeting constraint)

$x_1 \le 6,\ x_2 \ge 2,\ x_3 \ge 2$ (Advertisements constraint) $x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0$.]

11. A city hospital has the following minimal daily requirements for nurses.

Period	Clock Time (24 hr. day)	Minimal Number of Nurses Required
1	6 A.M. – 10 A.M.	2
2	10 A.M. – 2 P.M.	7
3	2 P.M. – 6 P.M.	15
4	6 P.M. – 10 P.M.	8
5	10 P.M. – 2 A.M.	20
6	2 A.M. – 6 A.M.	6

Nurses report to the hospital at the beginning of each period and work for 8 consecutive hours. The hospital wants to determine the minimum number of nurses to be employed so that there will be sufficient number of nurses available for each period. Formulate this as a linear programming problem by setting up appropriate constraints and objective function. Do not solve.

[**Hint.** Let $x_1, x_2, x_3, x_4, x_5, x_6$ be the number of nurses on duty at 6 *A.M.*, 10 *A.M.*, 2 *P.M.size5* , 6 *P.M.*, 10 *P.M.*, and 2 *A.M.*, respectively. Then the required LP formulation will be as follows :

Minimize $z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6$

subject to the constraints :

$$x_1 + x_2 \geq 7$$
$$x_2 + x_3 \geq 15$$
$$x_3 + x_4 \geq 8$$
$$x_4 + x_5 \geq 20$$
$$x_5 + x_6 \geq 6$$
$$x_6 + x_1 \geq 2$$
$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0].$$

4·3 GRAPHICAL SOLUTION OF TWO VARIABLE PROBLEMS

4.3–1 Graphical Procedure

Simple linear programming problems of two decision variables can be easily solved by *graphical method.* The outlines of graphical procedure are as follows :

Step 1. Consider each inequality–constraint as equation.

Step 2. Plot each equation on the graph, as each one will geometrically represent a straight line.

Step 3. Shade the feasible region. Every point on the line will satisfy the equation of the line. If the inequality-constraint corresponding to that line is ' ≤ ', then the region ***below*** the line ***lying*** in the first quadrant (due to non-negativity of variables) is shaded. For the inequality-constraint with ' ≥ ' sign, the region ***above*** the line in the first quadrant is shaded. The points lying in common region will satisfy all the constraints simultaneously. The common region thus obtained is called the ***feasible region.***

Step 4. Choose the convenient value of **z** (say = 0) and plot the objective function line.

Step 5. Pull the objective function line until the extreme points of the feasible region. In the maximization case, this line will stop farthest from the origin and passing through at least one corner of the feasible region. In the minimization case, this line will stop nearest to the origin and passing through at least one corner of the feasible region.

Step 6. Read the coordinates of the extreme point(s) selected in *Step 5*, and find the maximum or mininum (as the case may be) value of **z**. The following examples will make the outlined graphical procedure clear.

Q. 1. What is meant by linear programming problem ? Give brief description of the problem with illustrations. How the same can be solved graphically. What are the basic characteristics of a linear programming problem ? **[Meerut (Stat.) 98]**

2. Explain briefly the graphical method of solving linear programming problems. State its advantages and limitations.

4·3-2. Graphical Solution of Properly Behaved LP Problems

Example 26. *Find a geometrical interpretation and solution as well for the following LP problem :*

Maximize $z = 3x_1 + 5x_2$, *subject to restrictions :*

$x_1 + 2x_2 \leq 2000,\ x_1 + x_2 \leq 1500, x_2 \leq 600,$ *and* $x_1 \geq 0, x_2 \geq 0$.

Graphical Solution.

Step 1. ***(To graph the inequality-constraints).*** Consider two mutually perpendicular lines OX_1 and OX_2 as axes of coordinates. Obviously, any point (x_1, x_2) in the positive quadrant will certainly satisfy non-negativity restrictions : $x_1 \geq 0, x_2 \geq 0$. To plot the line $x_1 + 2x_2 = 2000$, put $x_2 = 0$, find $x_1 = 2000$ from this equation.

Then mark a point L such that $OL = 2000$ by assuming a suitable scale, say 500 units = 2 cm. Similarly, again put $x_1 = 0$ to find $x_2 = 1000$ and mark another point M such that $OM = 1000$.

Now join the points L and M. This line will represent the equation $x_1 + 2x_2 = 2000$ as shown in fig. 4.1.

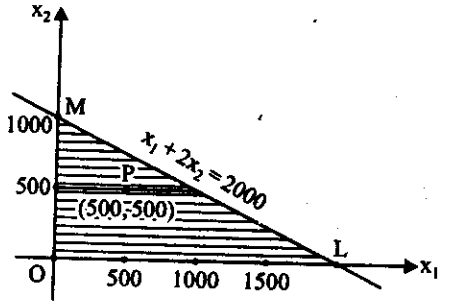


Fig. 4.1

Clearly, any point P lying on or below the line $x_1 + 2x_2 = 2000$ will satisfy the inequality $x_1 + 2x_2 \leq 2000$. (If we take a point (500, 500), *i.e.*, $x_1 = 500$, $x_2 = 500$, then we have $500 + 2 \times 500 < 2000$, which is true).

Similar procedure is now adoped to plot the other two lines : $x_1 + x_2 = 1500$ and $x_2 = 600$ as shown in the Figs. 4.2 and 4.3, respectively. Any point on or below the lines $x_1 + x_2 = 1500$ and $x_2 = 600$ will also satisfy other two inequalities : $x_1 + x_2 \leq 1500$, and $x_2 \leq 600$, respectively.

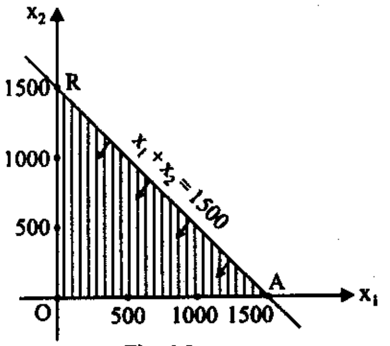


Fig. 4.2

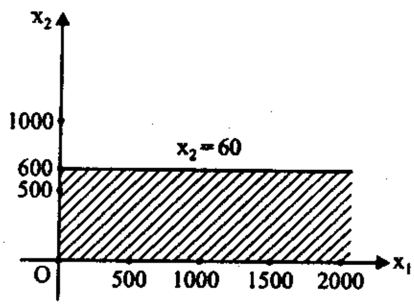


Fig. 4.3

Step 2. Find the *feasible region* or *solution space* by combining the Figs. 4.1, 4.2 and 4.3 together. A common shaded area *OABCD* is obtained (see Fig. 4.4) which is a set of points satisfying the inequality constraints :

$$x_1 + 2x_2 \leq 2000, x_1 + x_2 \leq 1500, x_2 \leq 600,$$

and non-negativity restrictions as $x_1 \geq 0, x_2 \geq 0$. ***Hence any point in the shaded area*** (including its boundary) is feasible solution to the given LPP.

Step 3. Find the co-ordinates of the corner points of feasible region O, A, B, C and D.

Step 4. Locate the corner point of optimal solution either by calculating the value of z for each corner point O, A, B, C, and D (or by adopting the following procedure).

Here, the problem is to find the point or points in the feasible region (collection of all feasible solutions) which maximize(s) the objective (or profit) function. For some fixed value of z, $z = 3x_1 + 5x_2$ is a straight line and *any point on it gives the same value of* z. Also, it should be noted that the lines corresponding to different values of z are parallel, because the gradient $(-3/5)$ of the line $z = 3x_1 + 5x_2$ remains the same throughout. For $z = 0$, *i.e.*, $0 = 3x_1 + 5x_2$, means a line which passes through the origin. To draw the line $3x_1 + 5x_2 = 0$, determine the ratio $\frac{x_1}{x_2} = \frac{-5}{3} = \frac{-500}{300}$.

Mark the point E moving 500 units distance from the origin on the negative side of X_1-axis. Then find the points F such that $EF = 300$ units in the positive direction of X_2-axis. Joining the point F and O, draw the line

$3x_1 + 5x_2 = 0$. Now go on drawing the lines parallel to this line until at least a line is found which is farthest from the origin but passes through at least one corner of the feasible region at which the maximum value of **z** is attained. It is also possible that such a line may coincide with one of the edge of feasible region. In that case, every point on that edge gives the maximum value of **z**.

In this example, maximum value of z is attained at the corner point B (1000, 500), which is the point of intersection of lines $x_1 + 2x_2 = 2000$ and $x_1 + x_2 = 1500$. Hence, the required solution is $x_1 = 1000, x_2 = 500$ and max. value **z** = Rs. 5500.

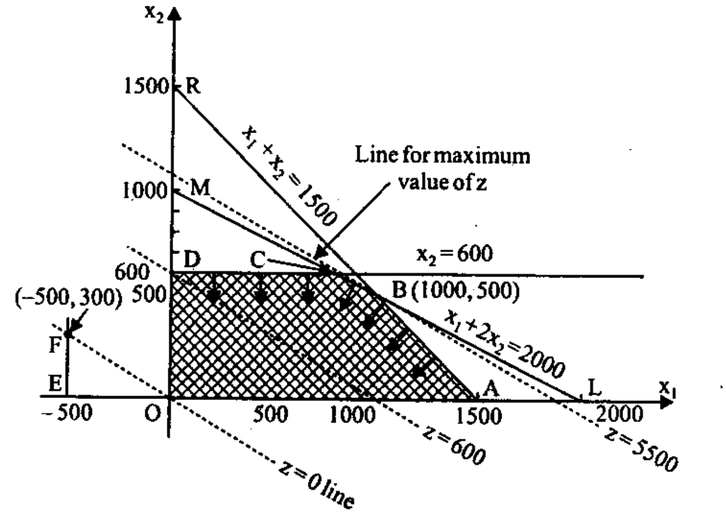


Fig. 4.4.

Note. If the number of vertices of feasible region is small, find the coordinates of vertices. As in above example, $O = (0, 0)$, $A = (1500, 0)$, $B = (1000, 500)$, $C = (800, 600)$, $D = (0, 600)$ are obtained by solving the pair of lines whose intersections are these points, respectively. The value of **z** corresponding to these points will be $z_0 = 0$, $z_A = 4500$, $z_B = 5500$, $z_C = 4500$, $z_D = 3000$. Clearly $z_B = 5500$ is maximum for the point $B (1000, 500)$ which gives the required solution.

Example 27. *Consider the problem*

$$\text{Max. } z = x_1 + x_2, \textit{ subject to,}$$
$$x_1 + 2x_2 \le 2000$$
$$x_1 + x_2 \le 1500$$
$$x_2 \le 600$$
and $$x_1, x_2 \ge 0.$$

Graphical Solution. This problem is of the same type as discussed earlier except the objective function is slightly changed. The feasible region will be similar to that of the above problem. Fig. 4.5 shows the objective function lines of the problem for three different values z_1, z_2, z_3 of **z**.

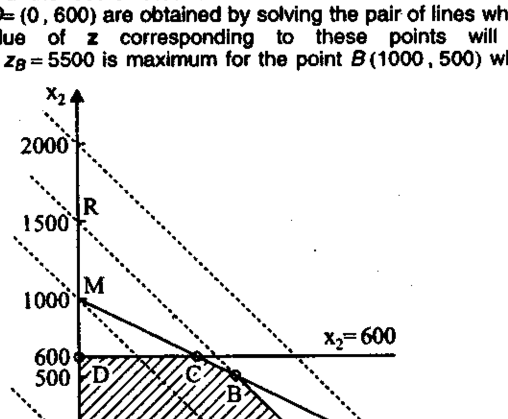


Fig. 4.5

It is clear from Fig. 4.5 that z_2 is the maximum value of **z**. It is quite interesting that the line z_2 representing the objective function lies along the edge *AB* of the polygon of feasible solutions. This indicates that the values of x_1 and x_2 which maximize **z** are not unique, but any point on the edge *AB* of *OABCD* the polygon will give the optimum value of **z**. The maximum value of **z** is always unique, but there will be an infinite number of feasible solutions which give unique value of **z**. Thus, two corners *A* and *B* as well as any point on the line *AB* (segment) give optimal solution of this problem.

It should be noted that if a linear programming problem has more than one optimum solution, there exists alternative optimum solutions. And, one of the optimum solutions will be corresponding to corner point *B*, *i.e.* $x_1 = 1000, x_2 = 500$ with max. profit **z** = Rs. 1500.

Example 28. *Solve the following LP problem graphically :*

$$\text{Max. } z = 8000x_1 + 7000x_2, \textit{ subject to}$$
$$3x_1 + x_2 \le 66, x_1 + x_2 \le 45, x_1 \le 20, x_2 \le 40 \textit{ and } x_1, x_2 \ge 0.$$

Solution. First, plot the lines $3x_1 + x_2 = 66$, $x_1 + x_2 = 45$, $x_1 = 20$ and $x_2 = 40$ and then shade the feasible region as shown in Fig. 4.6.

Draw a dotted line $8000x_1 + 7000x_2 = 0$ for $z = 0$ and continue to draw the lines till a point is obtained which is farthest from the origin but passing through at least one of the corners of the shaded (feasible) region. Fig. 4.6 shows that this point is P (10·5, 34·5) which is the point of intersection of lines

$$3x_1 + x_2 = 66 \text{ and } x_1 + x_2 = 45.$$

Hence, z is maximum for $x_1 = 10{\cdot}5$ and $x_2 = 34{\cdot}5$

$$\text{Max. } z = 8000 \times 10{\cdot}5 + 7000 \times 34{\cdot}5$$
$$= \text{Rs. } 325000. \quad \textbf{Ans.}$$

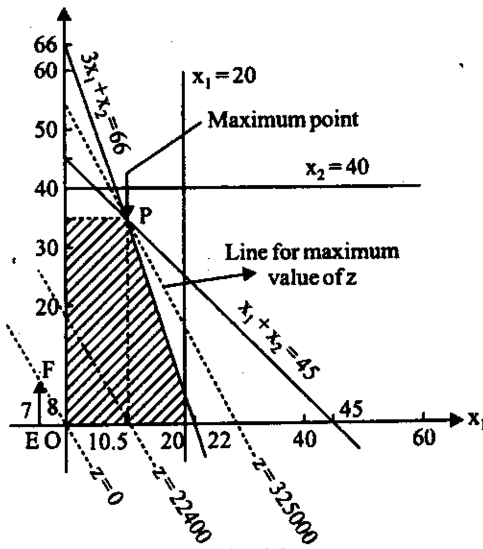


Fig. 4.6.

Example 29. *Old hens can be bought at Rs. 2 each and young ones at Rs. 5 each. The old hens lay 3 eggs per week and the young ones lay 5 eggs per week, each egg being worth 30 paise. A hen (young or old) costs Re. 1 per week to feed. I have only Rs. 80 to spend for hens, how many of each kind should I buy to give a profit of more than Rs. 6 per week, assuming that I cannot house more than 20 hens.* **[JNTU (B. Tech. 2002; Kota 93; Meerut (B.Sc.) 90, (M.Sc.) 90]**

Solution. *Formulation.* Let x_1 be the number of old hens and x_2 the number of young hens to be bought.

Since old hens lay 3 eggs per week and the yound ones lay 5 eggs per week, the total number of eggs obtained per week will be $= 3x_1 + 5x_2$.

Consequently, the cost of each egg being 30 paise, the total gain will be= Rs. $0{\cdot}30\,(3x_1 + 5x_2)$.

Total expenditure for feeding $(x_1 + x_2)$ hens at the rate of Re. 1 each will be = Rs. $1\,.(x_1 + x_2)$.

Thus, total profit **z** earned per week will be **z** = total gain − total expenditure

or $\quad \mathbf{z} = 0{\cdot}30\,(3x_1 + 5x_2) - (x_1 + x_2)$ or $\mathbf{z} = 0{\cdot}50\,x_2 - 0{\cdot}10\,x_1$ (objective function)

Since old hens can be bought at Rs. 2 each and young ones at Rs. 5 each and there are only Rs. 80 available for purchasing hens, the constraint is : $2x_1 + 5x_2 \le 80$.

Also, since it is not possible to house more then 20 hens at a time, $x_1 + x_2 \le 20$.

Also, since the profit is restricted to be more than Rs. 6, this means that the profit function **z** is to be maximized. Thus there is no need to add one more constraint, *i.e.* $0{\cdot}5x_2 - 0{\cdot}1\,x_1 \ge 6$.

Again, it is not possible to purchase negative quantity of hens, therefore $x_1 \ge 0$, $x_2 \ge 0$.

Finally, the problem becomes :
Find x_1 and x_2 (real numbers) so as to maximize the profit function $\mathbf{z} = 0{\cdot}50x_2 - 0{\cdot}10x_1$ subject to the constraints :
$2x_1 + 5x_2 \le 80$, $x_1 + x_2 \le 20$, and $x_1, x_2 \ge 0$.

Graphical Solution. Plot the straight lines $2x_1 + 5x_2 = 80$ and $x_1 + x_2 = 20$ on the graph and shade the feasible region as shown in fig 4.7.

The feasible region is $OBEC$. The coordinates of the extreme points of the feasible region are :

$O = (0, 0)$, $C = (20, 0)$, $B = (0, 16)$,
$E = (20/3, 40/3)$.

The values of z at these vertices are :

$z_o = 0$, $z_C = 0{\cdot}50 \times 0 - 0{\cdot}10 \times 20 = -2$,
$z_B = 0{\cdot}50 \times 16 - 0{\cdot}10 \times 0 = 8$,

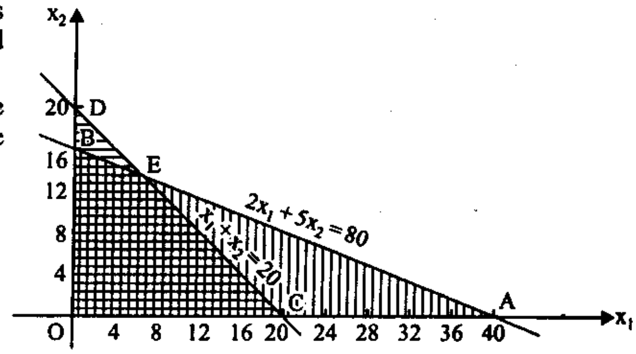


Fig. 4.7.

$$z_E = 0{\cdot}50 \times \frac{40}{3} - 0{\cdot}10 \times \frac{20}{3} = 6.$$

Since the maximum value of z is Rs. 8 which occurs at the point $B = (0, 16)$, the solution to the given problem is $x_1 = 0, x_2 = 16$, max. $z =$ Rs. 8.

Hence only 16 young hens I should buy in order to get the maximum profit of Rs. 8 (which is > 6).

Example 30. ***(Minimization problem)***

Consider the problem : Min. $z = 1{\cdot}5x_1 + 2{\cdot}5x_2$ *subject to* $x_1 + 3x_2 \geq 3, x_1 + x_2 \geq 2, x_1, x_2 \geq 0$.

Graphical Solution. The geometrical interpretation of the problem is given in Fig. 4.8. The minimum value of z is $z_A = 3{\cdot}5$. This minimum is attained at the point of intersection A of the lines $x_1 + 3x_2 = 3$ and $x_1 + x_2 = 2$. This is the unique point to give the minimum value of z. Now, solving these two equations simultaneously, the optimum solution is : $x_1 = 3/2, x_2 = 1/2$ and min. $z = 3{\cdot}5$.

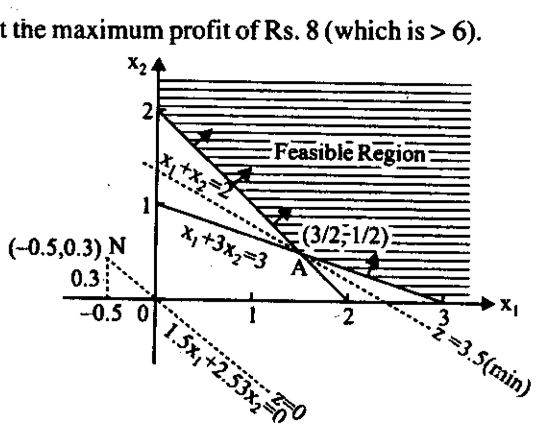


Fig. 4.8.

4·3–3 Graphical Solution in Some Exceptional Cases

The following examples show that there are certain exceptional cases which must be taken into consideration if a general technique for solving LP problems is to be developed.

Example 31. ***(Problem having unbounded solution)***

Max $z = 3x_1 + 2x_2$ *subject to* $x_1 - x_2 \leq 1, x_1 + x_2 \geq 3$, *and* $x_1, x_2 \geq 0$.

Graphical Solution. The region of feasible solutions is the shaded area in Fig. 4.9.

It is clear from this figure that the line representing the objective function can be moved far even parallel to itself in the direction of increasing z, and still have some points in the region of feasible solutions.

Hence z can be made arbitrarily large, and the problem has no finite maximum value of z. Such problems are said to have *unbounded solutions.*

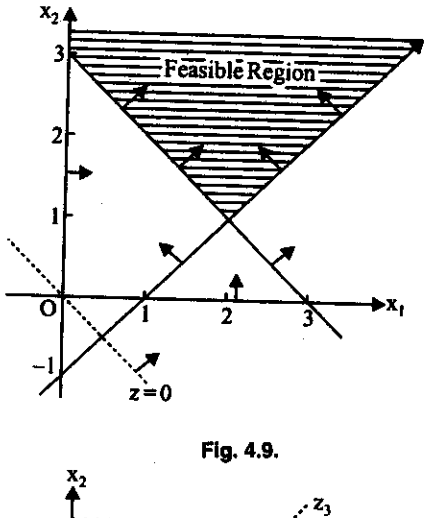


Fig. 4.9.

Infinite profit in practical problems of linear programming cannot be expected. If LP problem has been formulated by committing some mistake, it may lead to an unbounded solution.

Example 32. *Max.* $z = -3x_1 + 2x_2$ *subject to* $x_1 \leq 3$, $x_1 - x_2 \leq 0$, *and* $x_1, x_2 \geq 0$.

Graphical Solution. In *Example 31*, it has been seen that both the variables can be made arbitrarily large as z is increased. Here, an unbounded solution does not necessarily imply that all the variables can be made arbitrarily large as z approaches infinity. Here the variable x_1 remains constant as shown in Fig. 4.10.

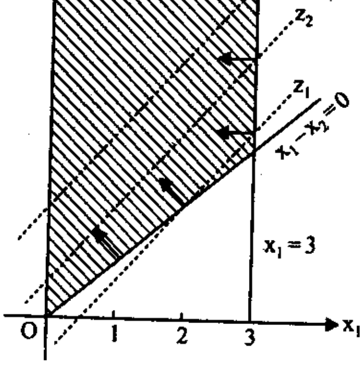


Fig. 4.10

Example 33. ***(Problem which is not completely normal)***

Maximize $z = -x_1 + 2x_2$ *subject to* $-x_1 + x_2 \leq 1, -x_1 + 2x_2 \leq 4,$ *and* $x_1, x_2 \geq 0$.

Graphical Solution. The problem is solved graphically in Fig. 4.11.

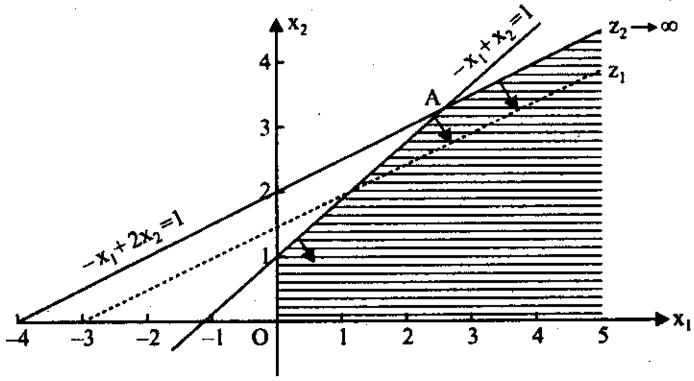


Fig. 4.11

Here the line of objective function coincides with the edge of Az_2 the region of feasible solutions. Thus, every point (x_1, x_2) lying on this edge $(-x_1 + 2x_2 = 4)$, which is going to infinity on the right gives $z = 4$, and is therefore an optimal solution.

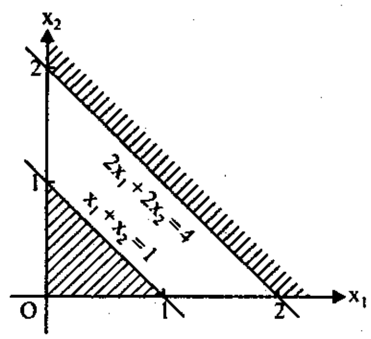


Fig. 4.12.

Example 34. ***(Problem with inconsistent system of constraints)***

Maximize $z = 3x_1 - 2x_2$ *subject to* $x_1 + x_2 \leq 1, 2x_1 + 2x_2 \geq 4,$ *and* $x_1, x_2 \geq 0$.

Graphical Solution. The problem is represented graphically in Fig. 4.12.

This figure shows that there is no point (x_1, x_2) which satisfies both the constraints simultaneously. Hence the problem has no solution because the constraints are inconsistent.

Example 35. ***(Constraints can be consistent and yet there may be no solution)***

Max. $z = x_1 + x_2$ *subject to* $x_1 - x_2 \geq 0, -3x_1 + x_2 \geq 3,$ *and* $x_1, x_2 \geq 0$.

Graphical Solution. Fig. 4.13 shows that there is no region of feasible solutions in this case. Hence there is no feasible solution. So the question of having optimal solution does not arise.

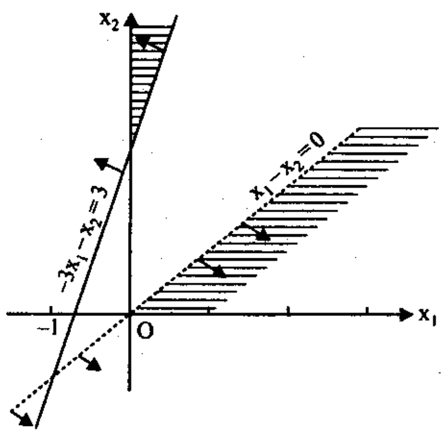


Fig. 4.13

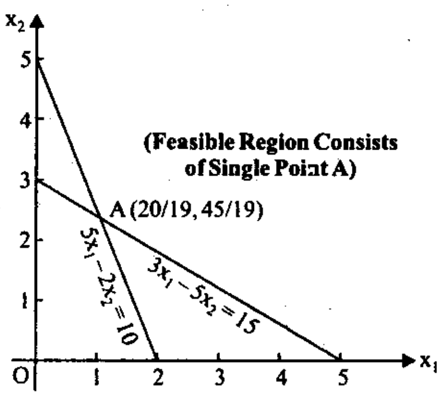


Fig. 4.14.

Example 36. ***(Problem in which constraints are equations rather than inequalities)***
Max. $z = 5x_1 + 3x_2$ *subject to* $3x_1 + 5x_2 = 15, 5x_1 + 2x_2 = 10, x_1 \geq 0, x_2 \geq 0$

Graphical Solution. Fig. 4.14 shows the graphical solution. Since there is only a single solution point A (20/19, 45/19), there is nothing to be maximized. Hence, a problem of this kind is of no importance. Such problems can arise only when the number of equations in the constratints is at least equal to the number of variables. If the solution is feasible, it is optimal. If it is not feasible, the problem has no solution.

Example 37. *A firm plans to purchase at least 200 quitals of scrap containing high quality metal X and low quality metal Y. It decides that the scrap to be purchased must contain at least 100 quintal of X-metal and not more than 35 quintals of Y-metal. The firm can purchase the scrap from two suppliers (A* and *B) in unlimited quantites. The percentage of X* and *Y metals in terms of weight in the scraps supplied by A* and *B is given below :*

Metals	Supplier A	Supplier B
X	25%	75%
Y	10%	20%

The price of A's scrap is Rs. 200 per quintal and that of B's is Rs. 400 per quintal. Formulate this problem as LP model and solve it to determine the quantities that the firm should buy from the two suppliers so as to minimize total purchase cost. **[Delhi (MBA) 98]**

Solution. The formulation of the given problem is :

Min. (total cost) $Z = 200x_1 + 400x_2$,

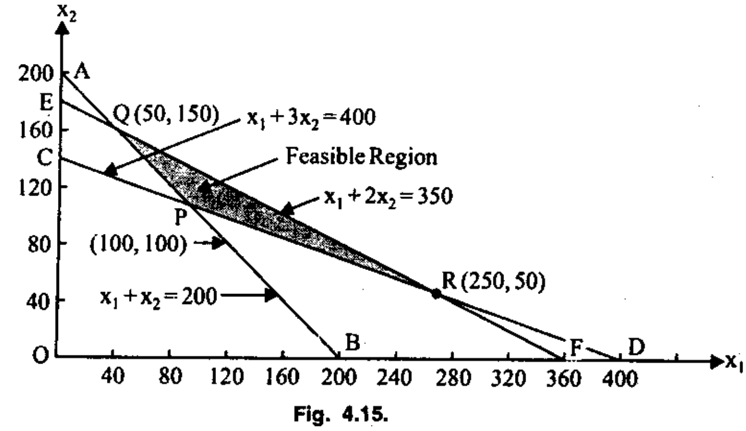


Fig. 4.15.

subject to the constraints :

$$x_1 + x_2 \geq 200, \frac{1}{4}x_1 + \frac{3}{4}x_2 \geq 100, \frac{1}{10}x_1 + \frac{1}{5}x_2 \leq 35, x_1 \geq 0, x_2 \geq 0.$$

where x_1, x_2 represent the number of quintals of scrap from two suppliers A and B respectively.

The feasible region is the shaded area PQR which is obtained by drawing the graph of the constraints :

$$x_1 + x_2 \geq 200, \ x_2 + 3x_2 \geq 400 \ \text{and} \ x_1 + 2x_2 \leq 175$$

The coordinates of the corner points of the feasible region are :

$$P\,(100, 100), Q\,(50, 150), R\,(250, 50)$$

The Z has the min. value at the point P (100, 100). Thus the answer is $x_1 = 100, x_2 = 100$, min. Z = Rs. 60,000.

Example 38. *The standard weight of a special purpose brick is 5 kg and it contains two basic ingredients* B_1 and B_2. B_1 *costs Rs. 5 per kg and* B_2 *costs Rs. 8 per kg. Strength considerations state that the brick contains not more than 4 kg of* B_1 *and minimum of 2 kg of* B_2. *Since the demand for the product is likely to be related to the price of the brick, find out graphically minimum cost of the brick satisfying the above conditions.*

Solution. The formulation of the given problem is :

Min (total cost) $Z = 5x_1 + 8x_2$,

subject to the constraints :

$x_1 \leq 4, x_2 \geq 2$ and $x_1 + x_2 = 5, x_1 \geq 0, x_2 \geq 0$, where $(x_1, x_2) =$ the amount of ingredients B_1 (in kg) and B_2 (in kg.) respectively. The given constraints are plotted on the graph as shown in the figure. It may be observed that feasible region has two corner points $P(3, 2)$ and $Q(4, 2)$. The minimum value of Z is found at $P(3, 2)$, *i.e.* $x_1 = 3, x_2 = 2$. Hence the optimum product mix is to have 3 kg. of ingredient B_1 and 2 kg. of ingredient B_2 of a special case brick in order to achieve the minimum cost of Rs. 31.

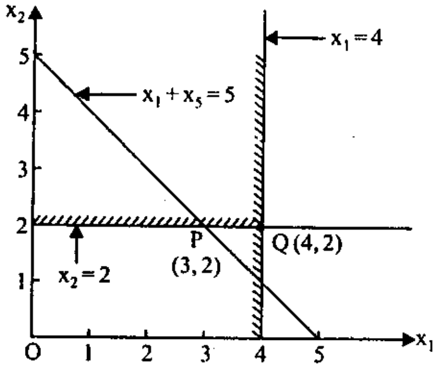


Fig. 4.16.

Example 39. *Assume that the following specify a generalized linear brogramming problem :*

$f(x) = 2x_1 + 1x_2$

subjec to $x_1 + x_2 \leq 6, x_1 \leq 3, 2x_1 + x_2 \geq 4, x_1, x_2 \geq 0.$

Graph this problem, identifying the three constraint equation lines and the feasible zone common to all of them. Plot dotted lines for values of 3, 6, 9 and 12 for the objective function $f(x)$ and for what values of x_1 and x_2 does it occur. **[I.E.S. (Mech.) 2001]**

Solution.

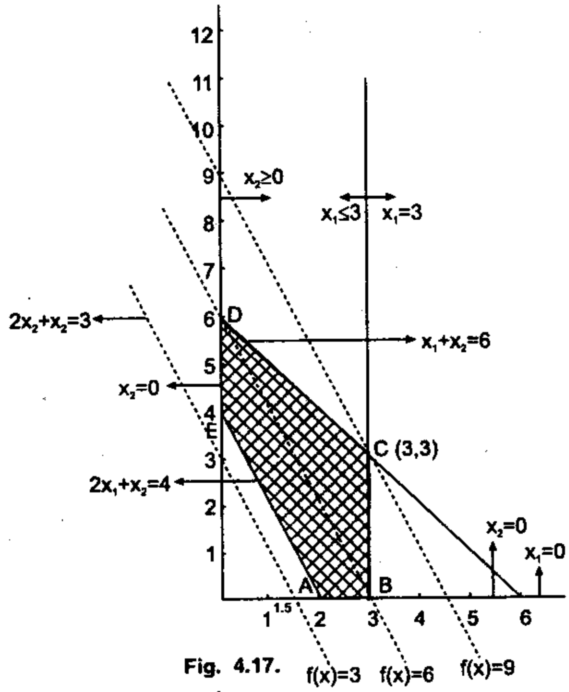


Fig. 4.17.

The whole shaded region would lie in the Ist quadrant

$f(x) = 3$

$\Rightarrow \quad 2x_1 + x_2 = 3 \longrightarrow$

$f(x) = 6$

$\Rightarrow \quad 2x_1 + x_2 = 6 \longrightarrow \quad \Rightarrow \quad$ By dotted lines

$f(x) = 9$

$\Rightarrow \quad 2x_1 + x_2 = 9 \longrightarrow$

$f(x) = 12$

$\Rightarrow \quad 2x_1 + x_2 = 12 \longrightarrow$

Basic feasible solutions are :

$A = (2, 0), B = (3, 0), C = (3, 3), D = (6, 0), E = (4, 0)$

$f(A) = 2 \times 2 + 1 \cdot 0 = 4, f(B) = 2 \times 3 + 1 \cdot 0 = 6, f(C) = 2 \times 3 + 1 \times 3 = 9, f(D) = 2 \times 4 + 1 \times 0 = 8.$

$\therefore f(C) = 9$ is maximum for $x_1 = 3, x_2 = 3$.

Application of LP on Management Accounts

Example 40. *A local travel agent is planning a charter trip to a major sea resort. The eight-day seven-night package includes the fare for round-trip travel, surface transportation, board and lodging and selected tour options. The charter trip is restricted to 200 persons and past experience indicates that there will not be any problem for getting 200 persons. The problem for the travel agent is to determine the number of Deluxe, Standard, and Economy tour packages to offer for this charter. These three plans each differ according to seating and service for the flight, quality of accomodation, meal plans and tour options. The following table summarizes the estimated prices for the three packages and the corresponding expenses for the travel agent. The travel agent has hired an air craft for the flat fee of Rs. 2,00,000 for the entire trip.*

Price and Costs for four packages per person

Tour plan	Price (Rs.)	Hotel costs (Rs.)	Meals & other expenses (Rs.)
Deluxe	10,000	3,000	4,750
Standard	7,000	2,200	2,500
Economy	6,500	1,900	2,200

In planning the trip, the following considerations must be taken into account :

(i) At least 10 per cent of the packages must be of the deluxe type.

(ii) At least 35 per cent but not more than 70 per cent must be of the standard type.

(iii) At least 30 per cent must be of the economy type.

(iv) The maximum number of deluxe packages available in any air craft is restricted to 60.

(v) The hotel desires that at least 120 of the tourists should be on the deluxe and standard packages together.

The travel agent wishes to determine the number of packages to offer in each type so as to maximize the total profit.

(a) Formulate the above as a linear programming problem.

(b) Restate the above linear programming problem in terms of two decision variables, taking advantage of the fact that 200 packages will be sold.

(c) Find the optimum solution using graphical methods for the restated linear programming problem and interpret your results. **[C.A. (May 91)]**

Solution. Let x_1, x_2, x_3 be the number of Deluxe, Standard & Economy tour packages restricted to 200 persons only to maximize the profits of the concern.

The contribution (per person) arising out of each type of tour package offered is as follows :

Package type offered	Price (Rs.)	Hotel Costs (Rs.)	Meals, etc. (Rs.)	Net profit (Rs.)
	(1)	(2)	(3)	(4) = (1) − [(2) + (3)]
Deluxe	10,000	3,000	4,750	2,250
Standard	7,000	2,200	2,500	2,300
Economy	6,500	1,900	2,200	2,400

Since the travel agent has to pay the flat fee of Rs. 2,00,000 for the chartered aircraft for the entire trip, the profit function will be :

$$\text{Max. } P = \text{Rs. } (2250x_1 + 2300x_2 + 2400x_3) - \text{Rs. } 2,00,000.$$

The constraints according to given conditions (i) to (v) are as follows :

$x_1 \geq 20$ from (i) $\quad x_3 \geq 60$ from (iii) $\quad x_1 + x_2 + x_3 = 200,$

$x_2 \geq 70$ from (ii) $\quad x_1 \leq 60$ from (iv)

$x_2 \leq 140$ from (ii) $\quad x_1 + x_2 \geq 120$ from (v) $\quad$ and $\quad x_1, x_2, x_3 \geq 0$

The compact form, above constraints can be reduced to the following forms :

$20 \leq x_1 \leq 60,\ 70 \leq x_2 \leq 140, x_3 \geq 60, x_1 + x_2 \geq 120, x_1 + x_2 + x_3 = 200$ *and* $x_1, x_2, x_3 \geq 0$, is

(a) The linear programming formation is as given above.

(b) Since $x_1 + x_1 + x_3 = 200$, *i.e.* $x_3 = 200 - (x_1 + x_2)$, substitute the value of x_3 in the above relations to get the following reduced LPP :

Max. $P = -150x_1 - 100x_2 + 2{,}80{,}000$, subject to $20 \le x_1 \le 60$, $70 \le x_2 \le 140$, $120 \le x_1 + x_2 \le 140$ and $x_1, x_2 \ge 0$.

(c) **Graphical Solution.** Refer to the following figure for the restated LP. problem in (b).

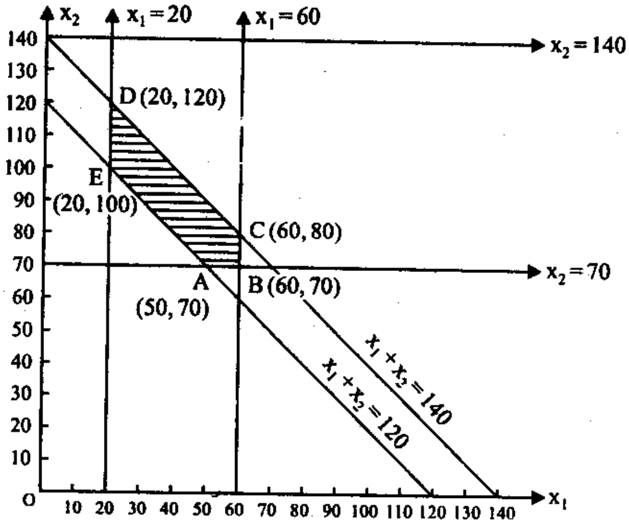


Fig. 4.18.

From above figure, we compute

Corner points	Coordinates of corner points	Values of objective function : $P = -150x_1 - 100x_2 + 2{,}80{,}000$
A	(50, 70)	P_A = Rs. 2,65,500
B	(60, 70)	P_B = Rs. 2,64,000
C	(60, 80)	P_C = Rs. 2,63,000
D	(20, 120)	P_D = Rs. 2,65,000
E	(20, 100)	P_E = Rs. 2,67,000

Thus maximum profit is attained at the corner point (20, 100).

Interpretation of Solution. Maximum profit of Rs. 2,67,000 is attained when $x_1 = 20, x_2 = 100$ and $x_3 = 200 - (x_1 + x_2) = 80$.

In other words, the travel agent should offer 20 *Delux*, 100 *Standard* and 80 *Economy* tour packages so as to get the maximum profit of Rs. 2,67,000.

Example 41. *Semicond is an electronics company manufacturing tape recorders and radios. Its per unit labour costs, raw material costs and selling prices are given in Table 1. An extract from its balance sheet on 31.3.1994 is shown in Table 2. Its current asset/current liability ratio (called the current ratio) is 2.*

Table 1 : *Cost Information*

For Products	Selling Price	Labour Cost	Raw Material Cost
Tape Recorder	Rs. 1,000	Rs. 500	Rs. 300
Radio	Rs. 900	Rs. 350	Rs. 400

Table 2 : *Extract from Balance Sheet as on 31.3.1994*

	Current Liabilities (Rs.)	Current Assets (Rs.)
Cash		1,00,000
* Accounts Receivable		30,000
** Inventory		70,000
Short-Term Bank Borrowing	1,00,000	

* *Accounts receivable is amount due from customers.*

** *100 units of raw material used for tape recorder and 100 units of raw material used for radio.*

Semicond must determine how many tape recorders and radios should be produced during April 94. Demand is large enough to ensure that all goods produced will be sold. All sales are on credit and payment for goods sold in April 94 will not be received until 31.5.94. During April 94, it will collect Rs. 20,000 in accounts receivable and it must payoff Rs. 10,000 of the outstanding short term bank borrowing and a monthly rent of Rs. 10,000. On 30.4.94, it will receive a shipment of raw material worth Rs. 20,000, which will be paid on May 31, 1994. The management has decided that the cash balance on April 30, 1994 must be at least 40,000. Also its banker requires that the current ratio as on April 30, 94 be at least 2. In order to maximize the contribution to profit for April 94 production it has to find the product mix for April 94. Assume that labour costs (wages) are paid in the month in which they are incurred. Formulate this as a linear programming problem and graphically solve it. **[C.A. (Nov. 94)]**

Solution. ***Formutation.*** Let x_1 and x_2 denote the number of units of tape recorders and radios respectively to be produced during April 1994.

Profit per unit of tape recorder = Selling price – (Labour cost + Raw material cost)
= Rs. 1000 – (Rs. 500 + Rs. 300) = Rs. 200

Similarly, profit per unit of radio = Rs. 900 – (Rs. 350 + Rs. 400) = Rs. 150

Company wishes to maximize its profit, therefore objective function is :

Maximize $P = 200x_1 + 150x_2$, subject to the following constraints :

(1) As per data given in the balanced sheet, the inventory available in the stock can be used only to produce 100 units of tape recorder and 100 units of radio. Therefore, $x_1 \leq 100$ and $x_2 \leq 100$.

(2) The management has decided that the cash balance on April 30, 1994 must be at least Rs. 40,000.

Cash balance = Cash in hand on March 31, 94 + Accounts receivable collected in April 94
– Bank borrowing paidoff in April – Monthly rent paid
– Labour cost paid during April 94.
= Rs. 1,00,000 + Rs. 20,000 – Rs. 10,000 – Rs. 10,000 – $(500x_1 + 350x_2)$

The management wants cash balance on (April 30, 1994) ≥ Rs. 40,000, *i.e.* ...(i)

Rs. 1,00,000 – $500x_1 - 350x_2 \geq$ Rs. 40,000

or Rs. 60,000 ≥ $500x_1 + 350x_2$ or $500x_1 + 350x_2 \leq$ Rs. 60,000 ...(ii)

(3) Bankers require that current ratio as on (April 30, 1994) ≥ 2,

Current ratio = current assets/current liabilities

Now we have to find the value of cash balance, accounts receivable, inventory and current liabilities as on April 30, 1994.

Cash balance = Rs. 1,00,000 – $500x_1 - 350x_2$. ...[from (i)]

Accounts receivables as on April 30, 1994 = Accounts receivable on March 31, 1994
+ Accounts receivable due from April sale – Accounts receivable collected during April
= Rs. 30,000 + $(1000x_1 + 900x_2)$ – Rs. 20,000 = Rs. 10,000 + $1000x_1 + 900x_2$

Inventory on April 30, 1994 = Inventory as on March 31, 1994 + Inventory received during April, 1994
– Inventory consumed during April, 1994
= Rs. 70,000 + Rs. 20,000 – $(300x_1 + 400x_2)$ = Rs. 90,000 – $(300x_1 + 400x_2)$

Current assets as on April 30, 1994 = Cash balance + Accounts receivables
+ Inventory value on April 30, 1994
= Rs. 1,00,000 – $500x_1 - 350x_2$ + Rs. 10,000 + $1000x_1 + 900x_2$
+ Rs. 90,000 – $300x_1 - 400x_2$
= Rs. 2,00,000 + $200x_1 + 150x_2$

Current liabilities as on April 30, 1994 = Value of bank borrowings as on March, 94
– Loan paid during April, 1994
+ Amount due on inventory received during April 1994

= Rs. 1,00,000 – Rs. 10,000 + Rs. 20,000 = Rs. 1,10,000.

But bank requires that current ratio as on April 30, 1994 be at least 2.

That is, current assets/current liabilities ≥ 2 or $\left(\frac{\text{Rs. } 2{,}00{,}000 + 200x_1 + 150x_2}{\text{Rs. } 1{,}10{,}000}\right) \geq 2$

or Rs. $2{,}00{,}000 + 200x_1 + 150x_2 \geq$ Rs. 2,20,000 or $200x_1 + 150x_2 \geq$ Rs. 20,000. ...(iii)

Thus the linear programming model for the Semicond is as follows :

Maximize $P = 200x_1 + 150x_2$, subject to the constraints :

$$x_1 \leq 100,\ x_2 \leq 100,\ 500x_1 + 350x_2 \leq 60000,\ 200x_1 + 150x_2 \geq 20000$$

Graphical Solution. The feasible region enclosed by the constraints is given by points A, B, C, D with coordinates :

$$A(25, 100),\ B(50, 100),\ C(100, \tfrac{200}{7}),\ D(100,0)$$

The profit at these coordinates is found below :

$A(25, 100)$: Rs. 200 × 25 + Rs. 150 × 100 = Rs. 20,000

$B(50, 100)$: Rs. 200 × 50 + Rs. 150 × 100 = Rs. 25,000

$C(100, 200/7)$: Rs. 200 × 100 + Rs. 150 × 200/7 = Rs. 24285·7

$D(100, 0)$: Rs. 200 × 100 + Rs. 150 × 0 = Rs. 20,000.

Since maximum profit is attained at the point $B(50, 100)$, Semicond can maximize its profit by producing 50 tap recorders and 100 radios during April, 1994 and the total profit contribution will be Rs. 25,000.

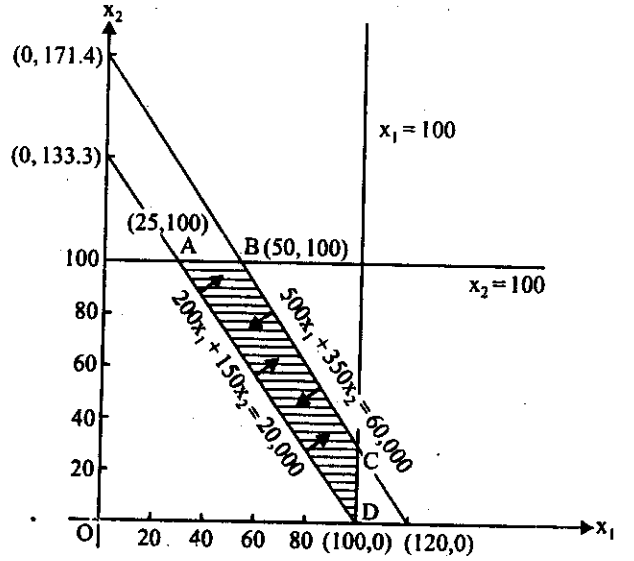


Fig. 4.19.

Q. Explain (i) No feasible solution, (ii) Unbounded solution. Give one example in each case.

4.3-4 Important Geometric Properties of LP Problems

Geometric properties of LP problems, which are observed while solving them graphically, are summarized below :

1. The region of feasible solutions has an important property which is called the *convexity property* in geometry, provided the feasible solution of the problem exists.

Convexity means that region of feasible solutions has *no holes* in them, that is, they are solids, and they have no cuts (like) on the boundary. This fact can be expressed more precisely by saying that the line joining any two points in the region also lies in the region.

2. The boundaries of the regions are lines or planes.

3. There are corners or exterme points on the boundary, and there are edges joining various corners.

4. The objective function can be represented by a line or a plane for any fixed value of z.

5. At least one corner of the region of feasible solutions will be an optimal solution whenever the maximum or minimum value of z is finite.

6. If the optimal solution is not unique, there are points other than corners that are optimal but in any case at least one corner is optimal.

7. The different situation is found when the objective function can be made arbitrarily large. Of course, no corner is optimal in that case.

EXAMINATION PROBLEMS

1. Solve the following LP problems by graphical method :

(a) Min. $z = 5x_1 - 2x_2$; s.t. $2x_1 + 3x_2 \geq 1$, $x_1, x_2 \geq 0$.
[**Hint.** Vertices of the feasible region are : $(\frac{1}{2}, 0)$, $(0, \frac{1}{3})$]
[**Ans.** $x_1 = 0$, $x_2 = 1/3$, min. $z = -2/3$].

(b) Max. $z = 5x_1 + 3x_2$; s.t. $3x_1 + 5x_2 \leq 15$, $5x_1 + 2x_2 \leq 10$; $x_1, x_2 \geq 0$
[**Hint.** Vertices of the feasible region are : (0, 0), (2, 0), (20/19, 45/19) and (0, 3).]
[**Ans.** $x_1 = 20/19$, $x_2 = 45/19$, max. $z = 235/19$]

(c) Max. $z = 2x_1 + 3x_2$; s.t. $x_1 + x_2 \leq 1$, $3x_1 + x_2 \leq 4$: $x_1, x_2 \geq 0$.
[**Hint.** Vertices of the feasible region are : (0, 0), (1, 0), (0, 1).]
[**Ans.** $x_1 = 0$, $x_2 = 1$, max. $z = 3$]

(d) Max. $z = 5x_1 + 7x_2$; s.t. $x_1 + x_2 \leq 4$, $3x_1 + 8x_2 \leq 24$, $10x_1 + 7x_2 \leq 35$, $x_1, x_2 \geq 0$. **[Meerut 90]**
[**Hint.** Vertices of the feasible region are : (0, 0), (7/2, 0), (7/3, 5/3), (8/5, 12/5) and (0, 3).]
[**Ans.** $x_1 = 8/5$, $x_2 = 12/5$, max. $z = 124/5$]

(e) Min. $z = -x_1 + 2x_2$; s.t. $-x_1 + 3x_2 \leq 10$, $x_1 + x_2 \leq 6$, $x_1 - x_2 \leq 2$, $x_1, x_2 \geq 0$.
[**Hint.** Vertices of the feasible region are : (0, 0) (2, 0), (4, 2), (2, 4) and (0, 10/3).]
[**Ans.** $x_1 = 2$, $x_2 = 0$, min. $z = -2$]

(f) Min. $z = 20x_1 + 10x_2$; s.t. $x_1 + 2x_2 \leq 40$, $3x_1 + x_2 \geq 30$, $4x_1 + 3x_2 \geq 60$, and $x_1 \geq 0$, $x_2 \geq 0$.
[**Hint.** Vertices of the feasible region are : (15, 0) (40, 0), (4, 18) and (6, 12).]
[**Ans.** $x_1 = 6$, $x_2 = 12$, min. $z = 240$].

(g) Max. $z = 3x_1 + 4x_2$; s.t. $x_1 - x_2 \leq -1$, $-x_1 + x_2 \leq 0$; $x_1, x_2 \geq 0$.
[**Ans.** The problem has no solution]

(h) Max. $z = 3x + 2y$; $-2x + 3y \leq 9$, $x - 5y \geq -20$; $x, y \geq 0$.
[**Ans.** The problem has an unbounded solution].

(i) Max. $z = x_1 + 3x_2$; $3x_1 + 6x_2 \leq 8$, $5x_1 + 2x_2 \leq 10$; $x_1, x_2 \geq 0$.
[**Hint.** The vertices of the feasible region are : (0, 0), (2, 0), (11/6, 5/12), (0, 4/3)
[**Ans.** $x_1 = 0$, $x_2 = 4/3$, max. $z = 4$]

(j) Max. $z = 7x_1 + 3x_2$; s.t. $x_1 + 2x_2 \geq 3$, $x_1 + x_2 \leq 4$, $0 \leq x_1 \leq 5/2$, $0 \leq x_2 \leq 3/2$. **[IPM (PGDBM) 2000]**
[**Hint.** The vertices of the feasible region are : (0, 0), (5/2, 1/4), (5/2, 3/2), and (0, 3/2).]
[**Ans.** $x_1 = 5/2$, $x_2 = 3/2$, max. $z = 22$].

(k) Max. $z = 3x + 4y$; s.t. $4x + 8y \leq 32$, $9x + 2y \geq 14$, $3x/2 + 5y \geq 15$, where $x, y \geq 0$.
[**Hint.** Vertices are : (3/4, 29/8), (2/3, 19/7), (0, 4), (0, 3).]
[**Ans.** $x = 3/4$, $y = 29/8$, max. $z = 17{\cdot}2$] **[JNTU (BE comp. Sc.) 2004]**

(l) Max. $z = 2x_1 + x_2$, s.t. $x_1 + 2x_2 \leq 10$, $x_1 + x_2 \leq 6$, $x_1 - x_2 \leq 2$, $x_1 - 2x_2 \leq 1$, $x_1, x_2 \geq 0$. **[IPM (PGDBM) 2000]**
[**Hint.** The vertices of the feasible region are : (0, 0), (1, 0), (3, 1), (4, 2), (2, 4) and (0, 5)]
[**Ans.** $x_1 = 4$, $x_2 = 2$, max. $z = 10$]

2. Does the following LPP has a feasible solution ? Max. $z = x_1 + x_2$, subject to $x_1 - x_2 \geq 0$, $3x_1 - x_2 \leq -3$. Show with the help of a graph.
[**Ans.** No feasible solution.]

3. Solve the following LPP's graphically :

(a) Maximize $z = 45x_1 + 80x_2$,
subject to the constraints :
$5x_1 + 20x_2 \leq 400$
$10x_1 + 15x_2 \leq 450$
$x_1, x_2 \geq 0$
[**Ans.** $x_1 = 24$, $x_2 = 14$, max. $z = 2200$]

(b) Minimize $z = 7y_1 + 8y_2$,
subject to the constraints :
$3y_1 + y_2 = 8$
$y_1 + 3y_2 \geq 11$
$y_1, y_2 \geq 0$
[**Ans.** $y_1 = 13/8$, $y_2 = 25/8$, min. $z = 291/8$]

(c) Maximize $z = 30x + 40y$.
subject to the constraints:
$50x + 36y \le 1{,}00{,}000$,
$25x + 36y \le 91{,}000$,
$x, y \ge 0$

[Ans. $x = 360$, $y = 2277.77$, max. $z = 1{,}01{,}911$]

(d) Minimize $z = -6x_1 - 4x_2$,
subject to the constraints:
$2x_1 + 3x_2 \ge 30$,
$3x_1 + 2x_2 \le 24$,
$x_1 + x_2 \ge 3$
$x_1, x_2 \ge 0$
[Ans. (i) $x_1 = 8$, $x_2 = 0$,
(ii) $x_1 = 12/5$, $x_2 = 42/5$, min. $z = -48$]

(e) Maximize $z = 3x_1 + 2x_2$,
subject to the constraints:
$2x_1 - x_2 \ge -2$
$x_1 + 2x_2 \ge 8$
$x_1, x_2 \ge 0$
[Ans. Unbounded solution]

(f) Maximize $z = 5x_1 + 7x_2$,
subject to the constraints:
$x_1 + x_2 \le 4$, $3x_1 + 8x_2 \le 24$
$10x_1 + 7x_2 \le 35$
$x_1, x_2 \ge 0$
[Ans. $x_1 = 8/5$, $x_2 = 12/5$, max. $z = 124/5$]

(g) Maximize $z = 120x_1 + 100x_2$,
subject to the constraints:
$10x_1 + 5x_2 \le 80$
$6x_1 + 6x_2 \le 66$
$4x_1 + 8x_2 \ge 24$
$5x_1 + 6x_2 \le 90$
$x_1, x_2 \ge 0$.
[Ans. $x_1 = 500$, $x_2 = 600$, max. $z = 1200$]

(h) Maximize $z = 2x_1 + 3x_2$,
subject to the constraints:
$x_1 + x_2 \ge 1$, $5x_1 - x_2 \ge 0$
$x_1 + x_2 \le 6$
$x_1 - 5x_2 \le 0$
$x_2 - x_1 \ge -1$
$x_2 \le 3$; $x_1, x_2 \ge 0$
[Ans. $x_1 = 4$, $x_2 = 3$, max. $z = 17$]

4. Find the maximum and minimum value of $z = 5x_1 + 3x_2$, subject to the constraints : $x_1 + x_2 \le 6$, $2x_1 + 3x_2 \ge 3$, $0 \le x_1 \ge 3$, $0 \le x_2 \ge 3$.
[Ans. $x_1 = 3$, $x_2 = 3$, min. $z = 24$].

5. Solve the LPP given below by graphical method and shade the region representing the feasible solution :
$x_1 - x_2 \ge 0$, $x_1 - 5x_2 \ge -5$, and $x_1, x_2 \ge 0$, minimum $z = 2x_1 - 10x_2$.
[Ans. $x_1 = 5/4$, $x_2 = 5/4$, min. $z = -10$]

6. Using graphical method find non-negative values of x_1 and x_2, which :
(a) Maximize $z = x_1 + 2x_2$, subject to the constraints:
$x_1 + x_2 \le 6$ $x_1 + x_2 \le 2$, $x_1 + 3x_2 \ge 6$, $-x_1 + 3x_2 \le 10$.
[Ans. $x_1 = 0$, $x_2 = 2$, max. $z = 4$]
(b) Minimize : $z = 600x_1 + 400x_2$,
subject to the constraints : $1500x_1 + 1500x_2 \ge 20{,}000$, $3000x_1 + 1000x_2 \ge 40{,}000$, $2000x_1 + 5000x_2 \ge 44{,}000$.
[Ans. $x_1 = 12$, $x_2 = 4$, min. $z = 8800$.]

7. Solve graphically the following LP problem :
Min. $z = 3x_1 + 5x_2$, s.t. $-3x_1 + 4x_2 \le 12$, $2x_1 - x_2 \ge -2$, $2x_1 + 3x_2 \ge 12$, $x_1 \le 4$, $x_2 \ge 2$, $x_1, x_2 \ge 0$ [Meerut 91]
[Ans. $x_1 = 3$, $x_2 = 2$, min. $z = 19$]

4.4 GENERAL FORMULATION OF LINEAR PROGRAMMING PROBLEM

The general formulation of the linear programming problem can be stated as follows :

In order to find the values of n decision variables $x_1, x_2, \ldots, x_n$ to maximize or minimize the objective function

$$z = c_1x_1 + c_2x_2 + c_3x_3 + \ldots + c_nx_n \qquad \ldots(4.7)$$

and also satisfy m-constraints :

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1j}x_j + \ldots + a_{1n}x_n \ (\le = \text{or} \ge)\ b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2j}x_j + \ldots + a_{2n}x_n \ (\le = \text{or} \ge)\ b_2 \\ \vdots \\ a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{ij}x_j + \ldots + a_{in}x_n \ (\le = \text{or} \ge)\ b_i \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mj}x_j + \ldots + a_{mn}x_n \ (\le = \text{or} \ge)\ b_m \end{array}\right\} \qquad \ldots(4.8)$$

where constraints may be in the from of any inequality ($\le$ or $\ge$) or even in the form of an equation (=), and finally satisfy the non-negativity restrictions

$$x_1 \geq 0, x_2 \geq 0, \ldots, x_j \geq 0, \ldots, x_n \geq 0. \quad \ldots(4.9)$$

However, by convention, the values of right side parameters b_i $(i = 1, 2, 3 \ldots, m)$ are restricted to non-negative values only. It is important to note that any negative b_i can be changed to a positive value on multiplying both sides of the constraint by -1. This will not only change the sign of all left side coefficients and right side parameters but will also change the direction of the inequality sign.

Q. 1. What do you mean by a L.P.P.? What are its limitations ?
2. Define a general linear programming problem. [Meerut (L.P.) 90]
3. What is linear programming problem (LPP) ? How can formulate a given problem into LPP ? [IGNOU 2001, 2000, 98, 97, 96]

4.5 SLACK AND SURPLUS VARIABLES [Jaunpur (B.Sc.) 96; Meerut 90]

1. Slack Variables. If a constraint has $\leq$ sign, then in order to make it an equality, we have to add something positive to the left hand side.

The non-negative variable which is added to the left hand side of the constraint to convert it into equation is called the slack variable.

For example, consider the constraints :

$$x_1 + x_2 \leq 2, 2x_1 + 4x_2 \leq 5, x_1, x_2 \geq 0 \quad \ldots(i)$$

We add the slack variables $x_3 \geq 0$, $x_4 \geq 0$ on the left hand sides of above inequalities respectively to obtain

$$\begin{aligned} x_1 + x_2 + x_3 \quad &= 2 \\ 2x_1 + 4x_2 \quad + x_4 &= 5 \\ x_1, x_2, x_3, x_4 &\geq 0. \end{aligned}$$

2. Surplus Variables. If a constraint has $\geq$ sign, then in order to make it an equality, we have to subtract something non-negative from its left hand side.

Thus the positive variable which is subtracted from the left hand side of the constraint to convert it into equation is called the surplus variable.

For example, consider the constraints :

$$x_1 + x_2 \geq 2, \; 2x_1 + 4x_2 \geq 5, \text{ and } x_1, x_2 \geq 0. \quad \ldots(ii)$$

We subtract the surplus variables $x_3 \geq 0$, $x_4 \geq 0$ from the left hand sides of above inequalities respectively to obtain

$$\begin{aligned} x_1 + x_2 - x_3 \quad &= 2 \\ 2x_1 + 4x_2 \quad - x_4 &= 5 \\ x_1, x_2, x_3, x_4 &\geq 0. \end{aligned}$$

4.6. STANDARD FORM OF LINEAR PROGRAMMING PROBLEM

The standard form of the linear programming problem is used to develop the procedure for solving general linear programming problem. The characteristics of the standard form are explained in the following steps :

Step 1. ***All the constraints should be converted to equations except for the non-negativity restrictions which remain as inequalities*** (≥ 0). Constraints of the inequality type can be changed to equations by augmenting (adding or subtracting) the left side of each such constraint by non-negative variables. These new variables are called ***slack variables*** and are added if the constraints are $(\leq)$ or subtracted if the constraints are $(\geq)$. Since in the case of $\geq$ constraint, the subtraced variable represents the surplus of the left side over the right side, it is common to refer to it as surplus variable. For convenience, however, the name 'slack' variable will also be used to represent this type of variable. In this respect, a surplus is regarded as a negative slack.

For example, consider the constraints : $3x_1 - 4x_2 \geq 7$, $x_1 + 2x_2 \leq 3$.

These constraints can be changed to equations by introducing slack variables x_3 and x_4 respectively. Thus, we get

$$3x_1 - 4x_2 - x_3 = 7, x_1 + 2x_2 + x_4 = 3, \text{ and } x_3 \geq 0, x_4 \geq 0.$$

Step 2. ***The right side element of each constraint should be made non-negative (if not).*** The right side can always be made positive on multiplying both sides of the resulting equation by (– 1) whenever it is necessary.

For example, consider the constraint as $3x_1 - 4x_2 \geq -4$

which can be written in the form of the equation $3x_1 - 4x_2 - x_3 = -4$

by introducing the surplus variable $x_3 \geq 0$.

Again, multiplying both sides by (-1), we get $-3x_1 + 4x_2 + x_3 = 4$ which is the constraint equation in standard form.

Step 3. ***All variables must have non-negative values.***

A variable which is *unrestricted in sign* (that is, positive, negative or zero) is equivalent to the difference between two non-negative variables. Thus, if x is unconstrained in sign, it can be replaced by $(x' - x'')$, where x' and x'' are both non-negative, that is, $x' \geq 0$ and $x'' \geq 0$.

Step 4. ***The objective function should be of maximization form.***

The minimization of a function $f(x)$ is equivalent to the maximization of the negative expression of this function, $f(x)$, that is,

$$\text{Min. } f(x) = -\text{Max } [-f(x)]$$

For example, the linear objective function

$$\text{Min. } z = c_1x_1 + c_2x_2 + \ldots + c_nx_n \qquad \ldots(4.10)$$

is equivalent to Max $(-z)$, *i.e.* Max $z' = -c_1x_1 - c_2x_2 - \ldots - c_nx_n$ with $z = -z'$.

Consequently, in any L.P problem, the objective function can be put in the maximization form.

Standard Form of General LPP with '≤' Constraints :

Now, applying above steps systematically to general form of L.P. problem with all (≤) constraints, the following standard form is obtained. Of course, no difficulty will arise to convert the general LPP with mixed constraints (≤ = ≥).

$$\text{Max. } z = c_1x_1 + c_2x_2 + \ldots + c_nx_n + 0x_{n+1} + \ldots + 0x_{n+m} \qquad \ldots(4.11)$$

subject to

$$\left.\begin{array}{llll} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n + x_{n+1} & & & = b_1 \\ a_{21}x_2 + a_{22}x_2 + \ldots + a_{2n}x_n & + x_{n+2} & & = b_2 \\ \vdots \qquad \vdots \qquad \vdots & & & \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n & & + x_{n+m} & = b_m \end{array}\right\} \qquad \ldots(4.12)$$

$$\text{where } \; x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0, x_{n+1} \geq 0, \ldots, x_{n+m} \geq 0. \qquad \ldots(4.13)$$

Note.

1. It should be remembered that the coefficient of slack variables $x_{n+1}, x_{n+2}, \ldots x_{n+m}$ in the objective function are assumed to be zero, so that the conversion of constraints to a system of simultaneous linear equations does not change the function to be optimized.
2. Since in the case of (≥ 0) constraints, the subtracted variable represents the surplus variable. However, the name slack variable may also represent this type. In this respect, a surplus is regarded as a negative slack.

Q. Define slack and surplus variables as involved in the L.P.P. How are these variables useful in solving a L.P.P.?

[AIMS (Bang.) MBA 2002]

Example 42. *Express the following L.P. problem in standard form.*

Min. $z = x_1 - 2x_2 + x_3$, *subject to* :

$2x_1 + 3x_2 + 4x_3 \geq -4$, $3x_1 + 5x_2 + 2x_3 \geq 7$, $x_1 \geq 0$, $x_2 \geq 0$ *and* x_3 *is unrestricted in sign.*

Solution. Proceeding according to above rules, the standard LP form becomes :

$$\text{Max } (z') = -x_1 + 2x_2 - (x_3' - x_3''), \text{ where } z' = -z, \text{ subject to}$$

$$-2x_1 - 3x_2 - 4(x_3' - x_3'') + x_4 \qquad = 4$$

$$3x_1 + 5x_2 + 2(x_3' - x_3'') \qquad - x_5 = 7$$

$$x_1 \geq 0, x_2 \geq 0, x_3' \geq 0, x_3'' \geq 0, x_4 \geq 0, x_5 \geq 0.$$

Of course, the number of variables will now increase to *six*.

4.7 MATRIX FORM OF LP PROBLEM

The linear programming problem in standard form [(3.11), (3.12), (3.13)] can be expressed in matrix form as follows :

Maximize $z = \mathbf{c}\mathbf{X}^T$ *(objective function)*

Subject to $\mathbf{AX} = \mathbf{b}, \mathbf{b} \geq \mathbf{0}$ *(constraint equation)*

$\mathbf{X} \geq \mathbf{0}$. *(non-negativity restriction)*

where $\mathbf{X} = (x_1, x_2, \ldots, x_n, x_{n+1}, \ldots, x_{n+m})$,

$\mathbf{c} = (c_1, c_2, \ldots, c_n, 0, 0, \ldots, 0)$, and $\mathbf{b} = (b_1, b_2, \ldots, b_m)$.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} & 1 & 0 & \ldots & 0 \\ a_{21} & a_{22} & \ldots & a_{2n} & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \ldots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & 0 & 0 & \ldots & 1 \end{bmatrix}$$

Similar treatment can be adopted in the case of mixed constraints ($\leq, =, \geq$). Following example will make this point clear.

The vector $\mathbf{x}$ is assumed to include all decision variables, (i.e. original, slack and surplus). For convenience, $\mathbf{x}$ is used to represent all types of variables. The vector $\mathbf{c}$ gives the corresponding coefficients in the objective function. For example, if the variable is slack, its corresponding coefficient will be zero.

Example 43. *Express the following LP problem in the matrix form.*

Max. $z = 2x_1 + 3x_2 + 4x_3$, *subject to*

$$x_1 + x_2 + x_3 \geq 5, x_1 + 2x_2 = 7, 5x_1 - 2x_2 + 3x_3 \leq 9, \text{ and } x_1 \geq 0, x_2 \geq 0, x_3 \geq 0.$$

Solution. This problem can be written in standard form as

$$\text{Max. } z = 2x_1 + 3x_2 + 4x_3 + 0x_4 + 0x_5 \text{ or Max. } z = (2, 3, 4, 0, 0)(x_1 x_2 x_3 x_4 x_5)^T$$

subject to $x_1 + x_2 + x_3 - x_4 = 5,\ x_1 + 2x_2 = 7, 5x_1 - 2x_2 + 3x_3 + x_5 = 9$

or

$$\begin{bmatrix} 1 & 1 & 1 & -1 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 5 & -2 & 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}$$

Therefore, $\mathbf{X} = (x_1\ x_2\ x_3\ x_4\ x_5)^T$, $\mathbf{c} = (2\ 3\ 4\ 0\ 0)$, $\mathbf{b} = (5\ 7\ 9)^T$, and

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & -1 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 5 & -2 & 3 & 0 & 1 \end{bmatrix}.$$

4.8 SOME IMPORTANT DEFINITIONS

Following are defined a few important terms for standard LPP (4.1, 4.2, 4.3) which are necessary to understand further discussion.

1. **Solution to LPP.** Any set $\mathbf{X} = \{x_1, x_2, \ldots, x_{n+m}\}$ of variables is called a ***solution*** to LP problem, if it satisfies the set of constraints (4.12) only.
2. **Feasible Solution (FS).** Any set $\mathbf{X} = \{x_1, x_2, \ldots, x_{n+m}\}$ of variables is called a ***feasible solution*** (or programme) of L.P. problem, if it satisfies the set of constraints (4.12) and non-negativity restrictions (4.13) also.
3. **Basic Solution (BS).** A ***basic solution*** to the set of constraints (4.12) is a *solution* obtained by setting any n variables (among $m+n$ variables) equal to zero and solving for remaining m variables, provided the determinant of the coefficients of these m variables is non-zero. Such m variables (of course, some of them may be zero) are called ***basic variables*** and remaining n zero-valued variables are called ***non-basic variables***. **[JNTU (B. Tech) 98]**

The number of basic solutions thus obtained will be at the most ${}^{m+n}C_m = \frac{(m+n)!}{n!\,m!}$, which is the number of combinations of $n+m$ things taken m at a time.

4. **Basic Feasible Solution (BFS).** A *basic feasible solution* is a *basic solution* which also satisfies the non-negativity restrictions (3.13), that is, all basic variables are non-negative.
Basic feasible solutions are of two types :
(a) ***Non-degenerate BFS.*** A non-degenerate basic feasible solution is the basic feasible solution which has exactly m positive x_i $(i = 1, 2, \ldots, m)$. In other words, all m basic variables are positive, and the remaining n variables will be all zero.
(b) ***Degenerate BFS.*** A basic feasible solution is called *degenerate,* if one or more basic variables are *zero-valued.*
5. **Optimum Basic Feasible Solution.** A basic feasible solution is said to be ***optimum,*** if it also optimizes (maximizes or minimizes) the objective function (4.11). **[Meerut (L.P) 90]**
6. **Unbounded Solution.** If the value of the objective function z can be increased or decreased indefinitely, such solutions are called ***unbounded solutions.*** **[Meerut 90]**

Note. Unless otherwise stated, solution means a feasible solution. However, an optimum solution to a linear programming problem imply that z has a finite maximum or finite minimum.

Q. 1. For the system $AX = b$ of m linear equations in n unknowns $(m < n)$ with rank$(A) = m$, define a basic solution. **[Meerut (IPM) 91]**
2. Explain the term optimal solution to a LPP. **[AIMS (Bangalore) MBA 2002]**
3. Define :
(i) Feasible Solution
(ii) Basic Solution **[JNTU (B. Tech.) 98]**
(iii) Basic Feasible Solution **[VTU (BE Mech.) 2002; Kanpur 96]**
(iv) Non-dgenerate BFS
(v) Degenerate BFS
(vi) Optimum Basic Feasible Solution **[Kanpur 96; Meerut (Stat.) 95 ; (Math.) 90]**
(vii) Unbounded Solution. **[JNTU (B. Tech) 98]**

Example 44. *Find all basic solutions for the system of simultaneous equations :*

$$2x_1 + 3x_2 + 4x_3 = 5 \quad \text{and} \quad 3x_1 + 4x_2 + 5x_3 = 6.$$

Solution. First decide the maximum number of basic solutions. The maximum possible number of basic solutions will be ${}^3C_2 = \frac{3!}{2!\,(3-2)!} = 3.$

Now, put $x_1 = 0$ and solve for x_2 and x_3. The values of x_2, x_3 thus obtained are : $x_2 = -1$, $x_3 = 2$.

Again, put $x_2 = 0$ and solve for x_1 and x_3. The values of x_1, x_3 are : $x_1 = -2$, $x_3 = \frac{3}{2}$.

Finally, put $x_3 = 0$ and solve for x_1 and x_2. The values of x_1, x_2 are : $x_1 = -2$. $x_2 = 3$.

This verifies that only three basic solutions exist which are *non-degenerate* and *infeasible* also.

EXAMINATION PROBLEMS

1. Determine all basic feasible solutions of the system of equations $2x_1 + x_2 + 4x_3 = 11$, $3x_1 + x_2 + 5x_3 = 14$.
[Ans. (i) $x_1 = 3$, $x_2 = 5$, $x_3 = 0$; (ii) $x_1 = 1/2$, $x_2 = 0$, $x_3 = 5/2$**]**
2. For the following system of linear equations, determine all the extreme points and their corresponding basic solutions:
$3x_1 + x_2 + 5x_3 + x_4 = 12$, $2x_1 + 4x_2 + x_3 + 2x_5 = 8$, where $x_1, x_2, x_3, x_4, x_5 \geq 0$.
3. Find all the basic solutions of the equations : $2x_1 + 6x_2 + 2x_3 + x_4 = 3$, $6x_1 + 4x_2 + 4x_3 + 6x_4 = 2$ **[Meerut (L.P.) 90]**
[Ans. $(0, 0, 2, -1)$, $(8/3, 0, 0, -7/3)$, $(0, 1/2, 0, 0)$, $(0, 1/2, 0, 0)$, $(0, 1/2, 0, 0)$ and $(-2, 0, 7/2, 0)$**]**
4. Show that the feasible solution $x_1 = 1$, $x_2 = 0$, $x_3 = 1$ and $z = 6$ to the system of equations : $x_1 + x_2 + x_3 = 2$, $x_1 - x_2 + x_3 = 2$, $x_j \geq 0$ $(j = 1, 2, 3)$ which minimize $z = 2x_1 + 3x_2 + 4x_3$, is not basic. **[Kanpur B.Sc. 95]**
5. Find all basic feasible solutions for the problem Max. $z = x_1 + 2x_2$ such that $x_1 + x_2 \leq 10$, $2x_1 - x_2 \leq 40$ and $x_1, x_2 \geq 0$.
[VTU (BE Mech.) 2002]

4.9 ASSUMPTIONS IN LINEAR PROGRAMMING PROBLEM

As examined from above examples, following are the assumptions in linear programming problem that limit its applicability.

(a) Proportionality. A primary requirement of linear programming problem is that the objective function and every constraint function must be ***linear.*** Roughly speaking, it simply means that if 1 kg of a product costs Rs. 2, then 10 kg will cost Rs. 20. If a steel mill can produce 200 tons in 1 hour, it can produce 1000 tons in 5 hours.

Intuitively, linearity implies that the product of variables such as $x_1 x_2$, powers of variables such as x_3^2, and combination of variables such as $a_1x_1 + a_2 \log x_2$, are not allowed.

(b) Additivity. As discussed in ***Example 1*** (page 54), additivity means if it takes t_1 hours on machine G to make product A and t_2 hours to make product B, then the time on machine G devoted to produce A and B both is $t_1 + t_2$, provided the time required to change the machine from product A to B is negligible.

The additivity may not hold, in general. If we mix several liquids of different chemical composition, then the total volume of the mixture may not be the sum of the volume of individual liquids.

(c) Multiplicativity. *It requires :*

(i) if it takes one hour to make a single item on a given machine, it will take 10 hours to make 10 such items; and

(ii) the total profit from selling a given number of units is the unit profit times the number of units sold.

(d) Divisibility. It means that the fractional levels of variables must be permissible besides integral values.

(e) Deterministic. All the parameters in the linear programming models are assumed to be known exactly. While in actual practice, production may depend upon chance also. Such type of problems, where some of the coefficients are not known, are discussed in the extension of sensitivity analysis known as parametric programming.

Significance of Assumptions :

A practical problem which completely satisfies all the above assumptions for linear programming is very rare indeed. Therefore, the user should be fully aware of the assumptions and appoximations involved and should satisfy himself that they are justified before proceeding to apply linear programming approach.

Q. State clearly the basic assumptions that are made in LPP.

4.10 LIMITATIONS OF LINEAR PROGRAMMING

Inspite of wide area of applications, some limitations are associated with linear programming techniques. These are stated below :

1. In some problems objective functions and constraints are not linear. Generally, in real life situations concerning business and industrial problems constraints are not linearly treated to variables.

2. There is no guarantee of getting integer valued solutions, for example, in finding out how many men and machines would be required to perform a particular job, rounding off the solution to the nearest integer will not give an optimal solution. Integer programming deals with such problems.

3. Linear programming model does not take into consideration the effect of time and uncertainty. Thus the model should be defined in such a way that any change due to internal as well as external factors can be incorporated.

4. Sometimes large-scale problems cannot be solved with linear programming techniques even when the computer facility is available. Such difficulty may be removed by decomposing the main problem into several small problems and then solving them separately.

5. Parameters appearing in the model are assumed to be constant. But, in real life situations they are neither constant nor deterministic.

6. Linear programming deals with only single objective, whereas in real life situations problems come across with multiobjectives. *Goal programming* and *multi-objective programming* deal with such problems.

Q. What are the limitations of linear programming technique ?

4.11 APPLICATIONS OF LINEAR PROGRAMMING

In this section, we discuss some important applications of linear programming in our life.

1. Personnel Assignment Problem. Suppose we are given m persons, n-jobs, and the expected productivity c_{ij} of ith person on the jth job. We want to find an assignment of persons $x_{ij} \geq 0$ for all i and j, to n jobs so that the average productivity of person assigned is maximum, subject to the conditions :

$$\sum_{j=1}^{n} x_{ij} \leq a_i \text{ and } \sum_{i=1}^{m} x_{ij} \leq b_j,$$

where a_i is the number of persons in personnel category i and b_j is the number of jobs in personnel category j. For details, refer the chaper of *Assignment Problems.*

2. Transportation Problem. We suppose that m factories (called sources) supply n warehouses (called destinations) with a certain product. Factory F_i ($i = 1\ 2, \ldots, m$) produces a_i units (total or per unit time), and warehouse W_j ($j = 1, 2, 3 \ldots, n$) requires b_j units. Suppose that the cost of shipping from factory F_i to warehouse W_j is directly proportional to the amount shipped; and that the unit cost is c_{ij}. Let the decision variables, x_{ij}, be the amount shipped from factory F_i to warehouse W_j. The objective is to determine the number of units transported from factory F_i to warehouse W_j so that the total transportation cost $\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$ is minimized. In the mean time, the supply and demand must be satisfied exactly.

Mathematically, this problem is to find x_{ij} ($i = 1, 2, \ldots, m; j = 1, 2, \ldots, n$) in order to minimize the total transportation cost $z = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} (c_{ij})$, subject to the restrictions of the form

$$\sum_{j=1}^{n} x_{ij} = a_i, \quad i = 1, 2, \ldots, m \text{ (factory)}$$

$$\sum_{i=1}^{m} x_{ij} = b_j, \quad j = 1, 2, \ldots, n \text{ (warehouse)}$$

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j, \text{ and } x_{ij} \geq 0, (i = 1, 2, \ldots, m; j = 1, 2, \ldots, n).$$

For detailed discussion, refer **chapter 10** on ***Transportation Problem.***

3. Efficiencing on Operation of System of Dams. In this problem, we determine variations in water storage of dams which generate power so as to maximize the energy obtained from the entire system. The physical limitations of storage appear as inequalities.

4. Optimum Estimation of Executive Compensation. The objective here is to determine a consistent plan of executive compensation in an industrial concern. Salary, job ranking and the amounts of each factor required on the ranked job level are taken into consideration by the constraints of linear programming.

5. Agricultural Applications. Linear programming can be applied in agricultural planning for allocating the limited resources such as acreage, labour, water, supply and working capital, etc. so as to maximize the net revenue.

6. Military Applications. These applications involve the problem of selecting an air weapon system against gurillas so as to keep them pinned down and simultaneously minimize the amount of aviation gasoline used, a variation of transportation problem that maximizes the total tonnage of bomb dropped on a set of targets, and the problem of community defence against disaster to find the number of defence units that should be used in the attack in order to provide the required level of protection at the lowest possible cost.

7. Production Management. Linear programming can be applied in production management for determining product mix, product smoothing, and assembly time-balancing.

8. Marketing Management. Linear programming helps in analysing the effectiveness of advertising compaign and time based on the available advertising media. It also helps travelling sales-man in finding the shortest route for his tour.

9. Manpower Management. Linear programming allows the personnel manager to analyse personnel policy combinations in terms of their appropriateness for maintaining a steady-state flow of people into through and out of the firm.

10. Physical Distribution. Linear programming determines the most economic and efficient manner of locating manufacturing plants and distribution centres for physical distribution.

Besids above, linear programming involves the applications in the area of administration, education, inventory control, fleet utilization, awarding contract, and capital budgeting etc.

Q. 1. Give a brief account of applications of linear programming problem.
2. Explain the meaning of a Linear Programming Problem stating its uses and give its limitations. [C.A. (May) 95]
3. State in brief uses of linear programming Technique.

4.12 ADVANTAGES OF LINEAR PROGARMMING TECHNIQUES

The advantages of linear programming techniques may be out-lined as follows:

1. Linear programming technique helps us in making the optimum utilization of productive resources. It also indicates how a decision maker can employ his productive factors most effectively by choosing and allocating these resources.
2. The quality of decisions may also be improved by linear programming techniques. The user of this technique becomes more objective and less subjective.
3. Linear programming technique provides practically applicable solutions since there might be other constraints operating outside the problem which must also be taken into consideration just because, so many units must be produced does not mean that all those can be sold. So the necessary modification of its mathematical solution is required for the sake of convenience to the decision maker.
4. In production processes, high lighting of bottlenecks is the most significant advantage of this technique. For example, when bottlenecks occur, some machines cannot meet the demand while others remain idle for some time.

Q. What are the advantages of Linear Programming Technique?

SELF EXAMINATION PROBLEMS

1. A manufacturer of Furniture makes two products : chairs and tables. Processing of these products is done on two machines A and B. A chair requires 2 hours on machine A and 6 hours on macine B. A table requires 5 hours on machine A and no time on machine B. There are 16 hours of time per day available on machine A and 30 hours on machine B. Profit gained by manufacturer from chair and a table is Re. 1 and Rs. 5 respectively. What should be daily production of each of the two products? [Bikaner 92 (S)]
[**Hint.** Solve graphically. Formulation is : Max. $z = x_1 + 5x_2$, s.t. $2x_1 + 5x_2 \leq 16$, $6x_1 \leq 30$, and $x_1\ x_2 \geq 0$.
Vertices of feasible region are : (0, 0), (6, 6/5), (0, 3·2)]
[**Ans.** 3·2 tables, no chair, max, profit = Rs. 16].
2. The ABC Electric Appliance Company produces two products : Refrigerators and ranges. Production takes place in two separate departments. Refrigerators are produced in Dapartment I and ranges are produced in Department II. The company's two products are produced and sold on a weekly basis. The weekly production cannot exceed 25 regrigerators in Department I and 35 ranges in Department II, because of the limited available facilities in these two departments. The company regularly employs a total of 60 workers in the two departments. A refrigerator requires 2 man-week of labour, while a range requires 1 man-week of labour. A refrigerator contributes a profit of Rs. 60 and a range contributes a profit of Rs. 40. How many units of refrigerators and ranges should the company produce to realize a maximum profit? [Delhi (M. Com.) 93]
[**Hint. Formulation** : Max. $z = 60x_1 + 40x_2$; s.t. $2x_1 + x_2 \leq 60$, $0 \leq x_1 \leq 25$, $0 \leq x_2 \leq 35$.
Vertices of the feasible region are : (0, 0), (25, 0), (25, 10), (25/2, 25), (0, 35)]
[**Ans.** 12·5 refrigerators, 35 ranges, max. profit z = Rs. 2150].

3. A farm is engaged in breeding pigs. The pigs are fed on various products grown on the farm. In view of the need to ensure certain nutrient constituents, it is necessary to buy products (call them A and B) in addition. The contents of the various products, per unit, in nutrients are vitamins, proteins etc. is given in the follwoing table :

Nutrients	Nutrient Content in		Min. Amount of Nutrient
	A	B	
M_1	36	6	108
M_2	3	12	36
M_3	20	10	100

The last column of the above table gives the minimum amount of nutrient constituents M_1, M_2, M_3 which must be given to the pigs. If the products A and B cost Rs. 20 and Rs. 4 per unit respectively, how much each of these two products should be bought so that the total cost is minimized ?

[**Hint.** ***Formulation*** **:** Min. $z = 20x_1 + 40x_2$, s.t. $36x_1 + 6x_2 \geq 108$, $3x_1 + 12x_2 \geq 36$, $20x_1 + 10x_2 \geq 100$; $x_1, x_2 \geq 0$]
[**Ans.** 4 units of product A, 2 units of product B, Min. Cost = Rs. 160].

4. A company produces two types of leather belts, say type A and B. Belt A is of superior quality and belt B is of a lower quality. Profits on the two types of belt are 40 and 30 paise per belt, respectively. Each belt of type A requires twice as much time as required by a belt of type B. If all belts were of type B, the company would produce 1,000 belts per day. But the supply of leather is sufficient only for 800 per day. Belt A requires a fancy buckle and 400 fancy buckles are available for this, per day. For belt of type B, only 700 buckles are available per day. How should the company manufacture the two types of belt in order to have maximum overall profit ? **[Jodhpur 93]**

[**Hint.** Formulation is : Max. $z = 0{\cdot}40x_1 + 0{\cdot}30x_2$, s.t. $x_1 + x_2 \leq 800$. $2x_1 + x_2 \leq 1000$, $0 \leq x_1 \leq 400$, $0 \leq x_2 \leq 700$.
Graphically. Vertices of feasible region are : (0, 0), (400, 0), (400, 200), (200, 600), (100, 700), (0, 700)].
[**Ans.** 200 belts of type A, 600 belts of type B, max. profit = Rs. 260].

5. A company sells two different products A and B. The company makes a profit of Rs. 40 and Rs. 30 per unit on products A and B respectively. The two products are produced in a common production process and are sold in two different markets. The production process has a capacity of 30,000 man-hours. It takes 3 hours to produce one unit of A and one hour to produce one unit of B. The market has been surveyed and company officials feel that the maximum number of units of A that can be sold is 8,000 and the maximum of B is 12,000 units. Subject to these limitations, the products can be sold in any convex combinations.
Formulate the above problem as a L.P.P. and solve it by graphical method.

[**Hint.** Formulation of the problem is : Max. $z = 40x_1 + 30x_2$, *s.t.*, $3x_1 + x_2 \leq 30{,}000$, $0 \leq x_1 \leq 8{,}000$, $0 \leq x_2 \leq 12{,}000$.
Vertices of the feasible region are : (0, 0), (8000, 0), (8000, 6000), (6000, 12000), (0, 12000)].
[**Ans.** $x_1 = 6000$ units of A, $x_2 = 12000$ units of B, max. z = Rs. 6,00,000].

6. A person require 10, 12 and 12 units of chemicals A, B and C respectively for his garden. A liquid product contains 5, 2 and one units of A, B and C respectively per jar. A dry product contains, 1, 2 and 4 units of A, B and C per carton. If the liquid product sells for Rs. 3 per jar and the dry product sells for Rs. 2 per carton, how many of each should be purchased to minimize the cost and meet the requirements. **[Banasthali (M.Sc.) 93]**

[**Hint.** Formulation : Min. $z = 3x_1 + 2x_2$, *s.t.* $5x_1 + x_2 \geq 10$, $2x_1 + 2x_2 \geq 12$, $x_1 + 4x_2 \geq 12$ and $x_1, x_2 \geq 0$.
The vertices of the feasible region are : (12, 0), (4, 2), (1, 5), (0, 10)].
[**Ans.** $x_1 = .1$ unit of liquid product, $x_2 = 5$ units of dry product, min. cost of Rs. 13].

7. An automobile manufacturer makes automobiles and trucks in a factory that is divided into two shops. Shop A, which perform the basic assembly operation must work 5 man-days on each truck but only 2 man-days on each automobile. Shop B, which performs finishing operations must work 3 man-days for each automobile or truck that it produces. Because of men and machine limitations shop A has 180 man-days per week available while shop B has 135 man-days per week. If the manufacturer makes a profit of Rs. 300 on each truck and Rs. 200 on each automobile, how many of each should be produced to maximize his profit ?

[**Hint.** Formulation is : Max. $z = 300x_1 + 200x_2$; *s.t.* $5x_1 + 2x_2 \leq 180$, $3x_1 + 3x_2 \leq 135$, $x_1, x_2 \geq 0$]
[**Ans.** $x_1 = 30$ trucks, $x_2 = 15$ automobiles per week, max. z = Rs. 12,000].

8. The manager of an oil refinary must decide on the optimal mix of two possible blanding processes of which the inputs and outputs per production run are as follows :

Process	Input (Units)		Output (Units)	
	Crude A	Crude B	Gasoline X	Gasoline Y
1	5	3	5	8
2	4	5	4	4

The maximum amounts available of crude A and B are 200 units and 150 units respectively. Market requirements show that at least 100 units of gasoline X and 80 units of gasoline Y must be produced. The profits per production run from process 1 and process 2 are Rs. 300 and Rs. 400 respectively. Solve the LP problem by graphical method. **[Gujarat (M.B.A) 1998]**

[**Hint.** Formulation is : Max. $z = 300x_1 + 400x_2$, *s.t.*

$5x_1 + 4x_2 \le 200$, $3x_1 + 5x_2 \le 150$, $5x_1 + 4x_2 \ge 100$, $8x_1 + 4x_2 \ge 80$, $x_1, x_2 \ge 0$.

The vertices of the feasible region are : (20, 0), (40, 0), (400/13, 150/13), (0, 30), (0, 25)].

[**Ans.** $x_1 = 400/13$, $x_2 = 150/13$, Max. $z = 1{,}80{,}000/13$].

9. A manufacturer makes two products P_1 and P_1 using two machines M_1 and M_2. Product P_1 requires 2 hours on machine M_1 and 6 hours on machine M_2. Product P_2 requires 5 hours on machine M_1 and no time on machine M_2. There are 16 hours of time per day available on machine M_1 and 30 hours on M_2. Profit margin from P_1 and P_2 is Rs. 2 and Rs. 10 per unit respectively. What should be the daily production mix to optimize profit?
[**Ans.** $P_1 = 3{\cdot}2$, $P_2 = 0$, max profit = Rs. 16.]

10. Upon completing the construction of his house Dr. Sharma discovers that 100 square feet of plywood scrap and 80 square feet of white-pine scrap are in usable form for the construction of tables and book cases. It takes 16 square feet of plywood and 8 square feet of white-pine to make a table; 12 square feet of plywood and 16 square feet of white-pine are required to construct a book case. By selling the finished products to a local furniture store, Dr. Sharma can realise a profit of Rs. 25 on each table and Rs. 290 on each book case. How may he most profitably use the left-over wood? Use graphical method to solve the problem.
[**Hint.** Formulation of this problem is :
Max. $z = 25x_1 + 20x_2$, subject to the constraints : $16x_1 + 12x_2 \le 100$, $8x_1 + 16x_2 \le 80$; $x_1, x_2 \ge 0$].
[**Ans.** 4 tables, 3 book-cases, max. profit = Rs. 160].

11. A television company has three major departments for manufacture of its two models, A and B. Monthly capacities are given as follows :

Departments	Per unit time requirements (hours)		Hours available this month
	Model A	Model B	
I	4·0	2·0	1600
II	2·5	1·0	1020
III	4·5	1·5	1600

The marginal profit of model A is Rs. 400 each and that of model B is Rs. 100 each. Assuming that the company can sell any quantity of either product due to favourable market conditions, determine the optimum out-put for both the models, the highest possible profit for this month and the slack time in the three departments.
[**Hint.** Formulation of the problem is : Max. $z = 400x_1 + 100x_2$, subject to
$4x_1 + 2x_2 \le 1600$, $2{\cdot}5x_1 + x_2 \le 1200$, $4{\cdot}5x_1 + 1{\cdot}5\,x_2 \le 1600$; $x_1, x_2 \ge 0$]
[**Ans.** $x_1 = 3200/9$, $x_2 = 0$, max. $z = 128000/9$].

12. A caterer knows that he will need 40 napkins on a given day and 70 napkins the day after. He can purchase napkins at 20 paise each and, after they are purchased, he can have dirty napkins laundered at 5 paise each for using the next dary. In order to minimize his costs, how many napkins should he purchase initially and how many dirty napkins should have laundered.
[**Hint.** Formulation of the problem is : Max. $z = 0{\cdot}20x_1 + 0{\cdot}05x_2$, subject to $x_1 \ge 70$, $x_2 \ge 40$, $x_1, x_2 \ge 0$]
[**Ans.** The caterer should buy 70 napkins and have 40 laundered after the first day].

13. The ABC company has been a producer of picture tubes to telivision sets and certain printed circuits for radio. The company has just expanded into full scale production and marketing of AM and AM–FM radio. It has built a new plant that can operate 48 hours per week. Production of an AM radio in the new plant will require 2 hours and production of an AM–FM radio will require 3 hours. Each AM radio will contribute Rs. 40/- to profits while an AM – FM radio will contribute Rs. 80/- to profits. The marketing department after extensive research, has determined that a maximum of 15 AM radios and, 10 AM–FM radios can be sold each week.
   (i) Formulate a linear programming model to determine the optimumn production mix of AM and AM–FM radio that will maximize profits.
   (ii) Solve the above problem using graphical method. **[Delhi (MBA) Nov. 98]**
[**Hint.** The formulation is : Max. $z = 40x_1 + 80x_2$, s.t $2x_1 + 3x_2 \le 48$, $x_1 \le 15$, $x_2 \ge 10$, $x_1 \ge 0$, $x_2 \ge 0$]
[**Ans.** $x_1 = 9$, $x_2 = 10$ and max $z = 1{,}160$]

14. A manufacturer produces two different models, X and Y of the same product. The raw materials r_1, r_2 are required for production. At least 18 kg of r_1, and 12 kg of r_2 must be used daily. Also, at most 34 hours of labour are to be utilised. 2 kg of r_1' are needed for each model X and 1 kg of r_1 for each model Y. For each model of X and Y, 1 kg of r_2 is required. It takes 3 hours to manufacture a model X and 2 hours to manufacture a model Y. The profit is Rs. 50 for each model X and Rs. 30 for each model Y. How many units of each model should be produced to maximize the profit.
[**Hint.** The formulation of the problem is; Max $z = 50x_1 + 30x_2$, st. $2x_1 + x_2 \ge 18$, $x_1 + x_2 \ge 12$, $3x_1 + 2x_2 \le 34$, $x_1 \ge 0$, $x_2 \ge 0$.
Ans. $x_1 = 10$, $x_2 = 2$ and max $z = 560$.]

15. The PQR Stone company sells stone secured from any of three adjacent quarries. The stone sold by the company must conform to the following specifications :
   Material X equal to 30%,
   Material Y equal to or less than 40%
   Material Z between 30% and 40%.

Stone from quarry A costs Rs. 10 per tonne and has the following properties :
Material X—20%; Material Y—60% and Material Z—20%.
Stone from quarry B costs Rs. 12 per tonne and has the following properties :
Material X—40%; Material Y—30% and Material Z—30%
Stone from quarry C costs Rs. 15 per tonne and has the following properties :
Material X—10%; Material Y—40% and Material Z—50%.

Formulate the above as a linear programming problem to minimize cost per tonne. **(Delhi (MBA) April 99)**

[**Hint.** The data can be summarized as follows :

	A	B	C	
X	20%	40%	10%	= 30%
Y	60%	30%	40%	≤ 40%
Z	30%	30%	50%	between 30% & 40%
Costs	10	12	15	

The problem is : Min. $z = 10x_1 + 12x_2 + 15x_3$

s.t. $2x_1 + 4x_2 + x_3 = 3,\ 6x_1 + 3x_2 + 5x_3 \le 4,\ 2x_1 + 3x_2 + 5x_3 \le 4$,

$2x_1 + 3x_2 + 5x_3 \ge 3\ ;\ x_1,\ x_2,\ x_3 \ge 0$.]

16. A firm produces three products A, B and C. It uses two types of raw materials I and II of which 5,000 and 7,500 units respectively are available. The raw material requirements per unit of the products are given below :

Raw Material	Requirement per unit of Product		
	A	B	C
I	3	4	5
II	5	4	5

The labour time for each unit of product A is twice that of product B and three times that of product C. The entire labour force of the firm can produce the equivalent of 3,000 units. The minimum demand of the three products is 600 units, 650 units, 500 units respectively. Also the ratios of the number of units produced must be equal to 2 : 3 : 4 assuming the profits per unit of A, B and C as Rs. 50, 50 and 80 respectively.

Formulate the problem, as a linear programming model in order to determine the number of units of each product which will maximize the profit. **(CA, Nov. 97)**

[**Hint.** Max. $z = 50x_1 + 50x_2 + 80x_3$, subject to

(i) $3x_1 + 4x_2 + 5x_3 \le 5{,}000$ and $5x_1 + 3x_2 + 5x_3 \le 7{,}500$ (ii) $x + \frac{1}{2}x_2 + \frac{1}{3}x_3 \le 3{,}000$

(iii) $x_1 \ge 600,\ x_2 \ge 650,\ x_3 \ge 500$ (iv) $\frac{1}{2}x_1 = \frac{1}{2}x_2$ and $\frac{1}{3}x_2 = \frac{1}{4}$, since ratio is 2 : 3 : 4.]

17. A cold-drink company has two bottling plants, located at two different places. Each plant produces three different drinks A, B and C. The capacities of two plants in number of bottles per day are as follows :

	Product A	Product B	Product C
Part I	3,000	1,000	2,000
Part II	1,000	1,000	6,000

A market survey indicates that during any particular month there will be a demand of 24,000 bottles of A; 16,000 bottles of B; and 48,000 bottles of C. The operating costs, per day, of running plants I and II are respectively 600 monetary units and 400 monetary units. How many days should the company run each plant during the month so that the production cost is minimised while still meeting the market demand ? Use Graphic method. **(Delhi (MFC) 96)**

[**Hint.** Min. $z = 600x_1 + 400x_2$, s.t. $3{,}000x_1 + 1{,}000x_2 \ge 24{,}000$

$10{,}000x_1 + 1{,}000x_2 \ge 16{,}000;\ 2{,}000x_1 + 6{,}000x_2 \ge 48{,}000;\ x_1 \ge 0,\ x_2 \ge 0$]

[**Ans.** $x_1 = 4,\ x_2 = 12$; min. $z = 7{,}200$.]

18. A company buying scrap metal has two types of scrap available to them. The first type of scrap metal has 20% of metal A, 10% of impurity and 20% of metal B by weight. The second type of scrap has 30% of metal A, 10% of impurity and 15% of metal B by weight. The company requires at least 120 kg. of metal A, at most 40 kg. of impurity and at least 90 kg. of metal B. The prices for the two scraps are Rs. 200 and Rs. 300 per kg. respectively. Determine the optimum quantities of the two scraps to be purchased so that the requirements of the two metals and the restriction on impurity are satisfied at minimum cost. **(Nagpur (MBA) 97)**

[**Hint.** Max. $z = 200x_1 + 300x_2$, s.t.

$0{\cdot}2x_1 + 0{\cdot}3x_2 \ge 120,\ 0{\cdot}1x_1 + 0{\cdot}1x_2 \le 40,\ 0{\cdot}5x_1 + 0{\cdot}15x_2 \ge 90;\ x_1 \ge 0,\ x_2 \ge 0$]

[**Ans.** No feasible solution.]

19. An oil refinery can blend three grades of crude oil to produce quality *R* and quality *S* petrol. Two possible blending processes are available. For each production run the older process uses 5 units of crude *A*, 7 units of crude *B* and 2 units of crude *C* to produce 9 units of *R* and 7 units of *S*. The newer process uses 3 units of crude *A*, 9 units of crude *B* and 4 units of crude *C* to produce 5 units of *R* and 9 units of *S* petrol. Because of prior contract commitments, the refinery must produce at least 500 units of *R* and at least 300 units of *S* for the next month. It has available 1500 units of crude *A*, 1900 units of crude *B* and 1000 units of crude *C*. For each unit of *R* the refinery receives Rs. 60 while for each unit of *S* it receives Rs. 90. Write down the linear programming formulation of the problem so as to maximize the revenue and solve by graphic method. **(Poona (MBA) Dec. 96)**

[**Hint.** Max. $z = 60(9x_1 + 5x_2) + 90(7x_1 + 9x_2)$, s.t.

$9x_1 + 5x_2 \geq 500,\ 7x_1 + 9x_2 \geq 300,\ 5x_1 + 3x_2 \leq 1{,}500,\ 7x_1 + 9x_2 \leq 1{,}900$

$2x_1 + 4x_2 \leq 1{,}000;\ x_1 \geq 0,\ x_2 \geq 0$]

20. What is redandant constraint ? What does it imply ? Does it affect the optimal solution to LPP ? **[VTU (BE Mech.) 2002; IGNOU 2000 (June)]**

CONVEX SETS

4.13 INTRODUCTION

So far we have derived geometrical properties from simple graphical examples of two dimensions. Now, we shall derive these properties, mathematically, for the general linear programming problem. In this chapter, we shall draw the conclusion that all the properties that hold true for simple problem (of two or three variables) also hold true for the general linear programming problem (of n variables), if we think of it as being represented graphically in an n-dimensional space.

First, we shall introduce a few important definitions and give proper names to the concepts that we have been using in our discussion. The main topic of this chapter is ***convex set theory***. Recently, however, the theory has found many important applications in *linear programming, games theory, economic and statistical decision theory.*

An optimal as well as feasible solution to an LP problem is obtained by choosing among several values of decision variables $x_1, x_2, \ldots x_n$ the one set of values that satisfy the given set of constraints simultaneously and also provide the optimal (most suitable) value of the given objective function.

Solution having values of decision variables x_j ($j = 1, 2, \ldots, n$) which satisfy the constraints of a general LP model is called the solution to that LP model.

Feasible Solution : Solution values of decision variables x_j ($j = 1, 2, \ldots, n$) which satisfy the constraints and non-negativity conditions of a general LP model are said to constitute the feasible solution to that LP model.

Basic Solution : For a set of m equations in n variables ($n > m$), a solution obtained by setting $(n - m)$ variables equal to zero and solving for remaining m equations in m variables is called a *basic solution.*

The $(n - m)$ variables whose value did not appear in this solution are called *non-basic variables* and the remaining m variables are called *basic variabels.*

While obtaining the optimal solution to the LP problem by the graphical method, the statement of the following theorems of linear programming is used—

(*i*) *The collection of all feasible solutions to an LP problem constitutes a convex set whose extreme points correspond to the basic feasible solution.*

(*ii*) *There are finite number of basic feasible solutions within the feasible solution space.*

(*iii*) *If the conves of the feasible solutions of the system* $\mathbf{Ax} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}$, *is a convex polyhedron, then at least one of the extreme points gives an optimal solution.*

Q. What is feasibility region ? Is it necessary that it should always be convex set ? **[IGNOU 2001]**

4.14 LINES AND HYPERPLANES

Lines. We consider a line in two-dimensional space as shown in Fig. 4·20. This line is passing through the points whose position vectors are $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$, and it is parallel to the vector $\mathbf{x}^{(2)} - \mathbf{x}^{(1)}$. Then the vector equation of the line joining $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ will be

$\mathbf{x} = \mathbf{x}^{(1)} + \lambda\,(\mathbf{x}^{(2)} - \mathbf{x}^{(1)})$, *i.e.* $\mathbf{x} = \lambda\,\mathbf{x}^{(2)} + (1-\lambda)\,\mathbf{x}^{(1)}$

Definition. *In R^n, the line through two distinct points $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ is defined to be the set of points :*

$X = \{\mathbf{x} : \mathbf{x} = \lambda\,\mathbf{x}^{(2)} + (1-\lambda)\,\mathbf{x}^{(1)}\}$, for all real λ.

Line segments. *In R^n, the line segment joining two points $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ is defined to be the set of points :*

$X = \{\mathbf{x} : \mathbf{x} = \lambda\,\mathbf{x}^{(2)} + (1-\lambda)\,\mathbf{x}^{(1)}\}$, for $0 \le \lambda \le 1$.

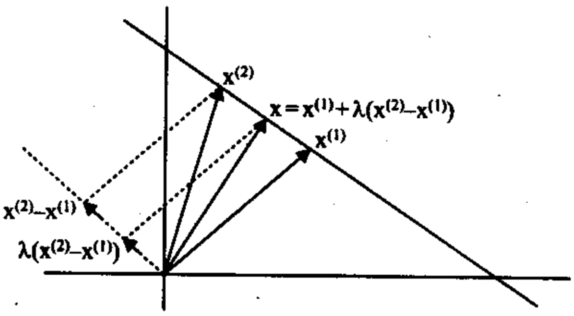


Fig. 4.20

Hyperplane. As the equation $c_1x_1 + c_2x_2 = z$ represents a line in R^2 for prescribed values of c_1, c_2, z; $c_1x_1 + c_2x_2 + c_3x_3 = z$ is the equation of the plane in R^3; in a similar fashion we can say that the set of points $\mathbf{x}$ satisfying

$$c_1x_1 + c_2x_2 + \ldots + c_nx_n = z, \text{ (not all } c_i = 0)$$

$$\text{or } \mathbf{cx} = z \text{ or } \mathbf{c}^T\mathbf{x} = z \text{ (in matrix form)} \qquad \ldots(4\cdot14)$$

defines a ***hyperplane*** for prescribed values of $c_1, c_2, \ldots, c_n$ and z. For optimum value of z, this hyperplane is called ***optimal hyperplane.***

A hyperplane $\mathbf{cx} = z$ in R^n divides whole of R^n into three mutually exclusive and collectively exhaustive sets. These are : *(i)* $X_1 = \{\mathbf{x} \mid \mathbf{cx} < z\}$ *(ii)* $X_2 = \{\mathbf{x} \mid \mathbf{cx} = z\}$ *(iii)* $X_3 = \{\mathbf{x} \mid \mathbf{cx} > z\}$.

Open half-spaces. *The sets $X_1 = \{\mathbf{x} \mid \mathbf{cx} < z\}$ and $X_3 = \{\mathbf{x} \mid \mathbf{cx} > z\}$ are called open half-spaces.*

Closed half-spaces. *The sets $X_4 = \{\mathbf{x} \mid \mathbf{cx} \le z\}$ and $X_5 = \{\mathbf{x} \mid \mathbf{cx} \ge z\}$ are called closed half-spaces.*

It can be easily seen that the hyperplanes are closed sets.

Supporting hyperplane. *Given a boundary point $\mathbf{w}$ of convex set X : then $\mathbf{cx} = z$ is called a* ***supporting hyperplane*** *at $\mathbf{w}$ if $\mathbf{cw} = z$ and if all of $\mathbf{x}$ lie in one closed half-space produced by the hyperplane, that is, $\mathbf{cu} \ge z$ for all $\mathbf{u} \in X$ or $\mathbf{cu} \le z$ for all $\mathbf{u} \in X$.*

4.15 CONVEX SET

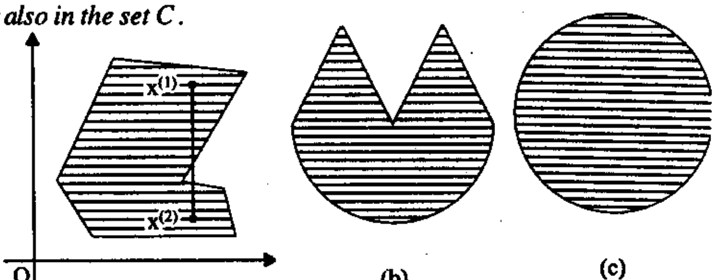
[IGNOU 98; Meerut 90]

Definition. *A set C in n–dimensional space is said to be* CONVEX *if for any two points $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$ in the set C, the line segment $[\mathbf{x}^{(1)} : \mathbf{x}^{(2)}]$ joining these points is also in the set C.*

Mathematically, this definition implies that $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are two distinct points in C, then every point $\mathbf{x} = \lambda\mathbf{x}^{(2)} + (1-\lambda)\,\mathbf{x}^{(1)}$, $0 \le \lambda \le 1$, must also be in the set C.

Symbolically, a subset $C \subset R^n$ is convex iff

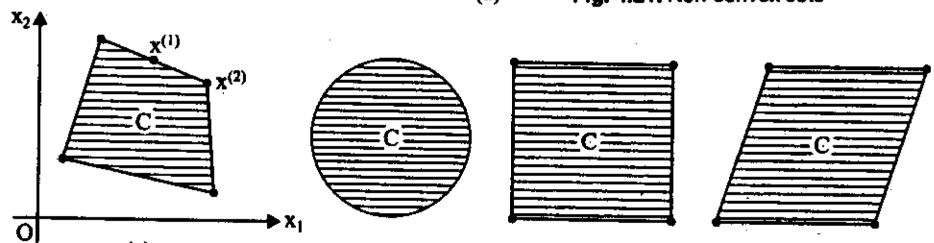
$$\mathbf{x}^{(1)}, \mathbf{x}^{(2)} \in C \Rightarrow [\mathbf{x}^{(1)} : \mathbf{x}^{(2)}] \subset C.$$



(a) (b) (c)

Fig. 4.21. Non-convex sets

It should also be noted that the set C containing only single point is convex, by



(a) (b) (c) (d)

Fig. 4.22 Convex sets

convention. Also the expression $\lambda \mathbf{x}^{(2)} + (1-\lambda)\mathbf{x}^{(1)}$ is called the convex combination of $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}$ for given λ.

To illustrate, the sets in Fig. 4.21 are ***non-convex*** while the sets in Fig. 4·22 are ***convex***. *The intersection of two convex sets is also a convex set.*

Q. 1. (a) Define a convex set. Show that the intersection of two convex sets is a convex set. **[Agra 98]**

(b) Show that $S = \{x_1, x_2, x_3 : 2x_1 - x_2 + x_3 \le 4\} \subset R^3$ is a convex set. **[Meerut (L.P.) 90]**

2. Define convex set. Prove that the set $S = \{(x_1, x_2) \mid x_1^2 + x_2^2 \le 4\}$ is a convex set. **[Virbhadrah 2000; IGNOU (MCA II) 98]**

Illustrative Examples

Example 1. *Show that* $C = \{(x_1, x_2) : 2x_1 + 3x_2 = 7\} \subset R^2$ *is a convex set.* **[Meerut (MA) 93 (P)]**

Solution. Let $\mathbf{X}, \mathbf{Y} \in C$, where $\mathbf{X} = (x_1, x_2)$, $\mathbf{Y} = (y_1, y_2)$.

The line segment joining $\mathbf{X}$ and $\mathbf{Y}$ is the set

$$W = \{\mathbf{W} : \mathbf{W} = \lambda\mathbf{X} + (1-\lambda)\mathbf{Y}, 0 \le \lambda \le 1\}$$

For some λ, $0 \le \lambda \le 1$, let $\mathbf{W} = (w_1, w_2)$ be a point of set W, so that

$$w_1 = \lambda x_1 + (1-\lambda) y_1,\ w_2 = \lambda x_2 + (1-\lambda) y_2$$

Since $\mathbf{X}, \mathbf{Y} \in C$, $2x_1 + 3x_2 = 7$ and $2y_1 + 3y_2 = 7$.

But,

$$\begin{aligned} 2w_1 + 3w_2 &= 2[\lambda x_1 + (1-\lambda)y_1] + 3[\lambda x_2 + (1-\lambda)y_2] \\ &= \lambda[2x_1 + 3x_2] + (1-\lambda)[2y_1 + 3y_2] \\ &= \lambda . 7 + (1-\lambda) . 7 = 7 \end{aligned}$$

Therefore $\mathbf{W} = (w_1, w_2) \in C$

Since $\mathbf{W}$ is any point of C, $\mathbf{X}, \mathbf{Y} \in C \Rightarrow [\mathbf{X} : \mathbf{Y}] \subset C$.

Hence C is convex.

Example 2. *For any points* $\mathbf{X}, \mathbf{Y} \in R^n$, *show that the line segment* $[\mathbf{X} : \mathbf{Y}]$ *is a convex set.*

Solution. Let $\mathbf{U}, \mathbf{V} \in [\mathbf{X} : \mathbf{Y}]$, so that

$$\left.\begin{aligned} \mathbf{U} &= t\mathbf{X} + (1-t)\mathbf{Y},\ 0 \le t \le 1 \\ \text{and}\quad \mathbf{V} &= s\mathbf{X} + (1-s)\mathbf{Y},\ 0 \le s \le 1 \end{aligned}\right\} \quad ...(1)$$

Now let $\mathbf{W}$ be a point of line segment $[\mathbf{U} : \mathbf{V}]$, so that

$$\mathbf{W} = \lambda\mathbf{U} + (1-\lambda)\mathbf{V},\ 0 \le \lambda \le 1. \quad ...(2)$$

From (1) and (2), we have $\mathbf{W} = \{t\lambda + s(1-\lambda)\}\mathbf{X} + \{\lambda(1-t) + (1-\lambda)(1-s)\}\mathbf{Y}$.

If we set, $\mu = t\lambda + s(1-\lambda)$, then $1 - \mu = 1 - t\lambda - s(1-\lambda)$

$$= \lambda + (1-\lambda) - t\lambda - (1-\lambda)s = (1-t)\lambda + (1-\lambda)(1-s)$$

Since $0 \le t \le 1, 0 \le s \le 1 \Rightarrow 0 \le t\lambda + s(1-\lambda) \le 1 \Rightarrow 0 \le \mu \le 1$, therefore

$$\mathbf{W} = \mu\mathbf{X} + (1-\mu)\mathbf{Y}, 0 \le \mu \le 1 \Rightarrow \mathbf{W} \in [\mathbf{X} : \mathbf{Y}].$$

Since $\mathbf{W}$ is any point of $[\mathbf{U} : \mathbf{V}]$, we have $[\mathbf{U} : \mathbf{V}] \subset [\mathbf{X} : \mathbf{Y}]$

$$\therefore \quad \mathbf{U}, \mathbf{V} \in [\mathbf{X} : \mathbf{Y}] \Rightarrow [\mathbf{U} : \mathbf{V}] \subset [\mathbf{X} : \mathbf{Y}].$$

Hence $[\mathbf{X} : \mathbf{Y}]$ is a convex set.

Example 3. *Show that* $S = \{(x_1, x_2, x_3) : 2x_1 - x_2 + x_3 \le 4\} \subset R^3$, *is a convex set.* **[Meerut 92, 90]**

Solution. Let $\mathbf{X} = (x_1, x_2, x_3)$ and $\mathbf{Y} = (y_1, y_2, y_3)$ be two points of S. Then, by the given condition

$$2x_1 - x_2 + x_3 \le 4 \text{ and } 2y_1 - y_2 + y_3 \le 4 \quad ...(1)$$

Now let $\mathbf{W} = (w_1, w_2, w_3)$ be any point of $[\mathbf{X} : \mathbf{Y}]$ so that $0 \le \lambda \le 1$,

$$\therefore \quad w_1 = \lambda x_1 + (1-\lambda) y_1,\ w_2 = \lambda x_2 + (1-\lambda) y_2,\ w_3 = \lambda x_3 + (1-\lambda) y_3 \quad ...(2)$$

From (1) and (2), we have

$$\begin{aligned} 2w_1 - w_2 + w_3 &= \lambda(2x_1 - x_2 + x_3) + (1-\lambda)(2y_1 - y_2 + y_3) \\ &\le 4\lambda + 4(1-\lambda) = 4 \end{aligned}$$

$\therefore$ $\mathbf{W} = (w_1, w_2, w_3)$ is a point of S. Thus, $\mathbf{X}, \mathbf{Y} \in S \Rightarrow [\mathbf{X} : \mathbf{Y}] \subset S$

Hence S is convex.

Example 4. *Show that in* R^3, *the closed ball* $x_1^2 + x_2^2 + x_3^2 \le 1$, *is a convex set.* **[Meerut (M.Sc.) 93]**

Solution. Let $S = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 \le 1\}$.

Also, let $\mathbf{X}, \mathbf{Y} \in S$, where $\mathbf{X} = (x_1, x_2, x_3)$, $\mathbf{Y} = (y_1, y_2, y_3)$.

Then, by the given condition, we have

$$x_1^2 + x_2^2 + x_3^2 \le 1 \text{ and } y_1^2 + y_2^2 + y_3^2 \le 1 \quad ...(1)$$

Now, for some scalar λ, $0 \le \lambda \le 1$, we have

$$\| \lambda \mathbf{X} + (1-\lambda)\mathbf{Y} \|^2 = [\lambda x_1 + (1-\lambda)y_1]^2 + [\lambda x_2 + (1-\lambda)y_2]^2 + [\lambda x_3 + (1-\lambda)\, y_3]^2$$

$$= \lambda^2 (x_1^2 + x_2^2 + x_3^2) + (1-\lambda)^2 [y_1^2 + y_2^2 + y_3^2] + 2\lambda (1-\lambda)\, [x_1y_1 + x_2y_2 + x_3y_3]$$

By *Schwartz's inequality*,

$$(x_1y_1 + x_2y_2 + x_3y_3) \le \sqrt{x_1^2 + x_2^2 + x_3^2}\, \sqrt{y_1^2 + y_2^2 + y_3^2}$$

Using (1), we have

$$\|\lambda \mathbf{X} + (1-\lambda)\mathbf{Y}\|^2 \le \lambda^2 + (1-\lambda)^2 + 2\lambda(1-\lambda) = [\lambda + (1-\lambda)]^2$$
$$\le 1$$

$\therefore$ $\lambda \mathbf{X} + (1-\lambda)\mathbf{Y}$ is a point of S. Thus, $\mathbf{X}, \mathbf{Y} \in S \Rightarrow [\mathbf{X} : \mathbf{Y}] \subset S$. Hence S is convex.

4·15-1 Some Important Theorems

Theorem 4.1. *A hyperplane in R^n is a convex set.*

Proof. Let $\mathbf{c}\mathbf{x} = z$ be a hyperplane and also let $\mathbf{x}_1$ and $\mathbf{x}_2$ are any two points on the hyperplane. Then,

$$\mathbf{c}\mathbf{x}_1 = z \text{ and } \mathbf{c}\mathbf{x}_2 = z\,.$$

Therefore, for $0 \le \lambda \le 1$,

$$\mathbf{c}\,[\lambda \mathbf{x}_1 + (1-\lambda)\,\mathbf{x}_2] = \mathbf{c}\,(\lambda \mathbf{x}_1) + \mathbf{c}\,[(1-\lambda)\,\mathbf{x}_2] = \lambda\,(\mathbf{c}\mathbf{x}_1) + (1-\lambda)\,\mathbf{c}\mathbf{x}_2 = \lambda z + (1-\lambda)\, z = z$$

Hence the point $\lambda \mathbf{x}_1 + (1-\lambda)\,\mathbf{x}_2$, for $0 \le \lambda \le 1$ lies in the hyperplane. So the hyperplane is convex.

Theorem 4.2. *The closed half spaces $H_1 = \{\mathbf{x} \mid \mathbf{c}\mathbf{x} \ge z\}$ and $H_2 = \{\mathbf{x} \mid \mathbf{c}\mathbf{x} \le z\}$ are convex sets.*

[Meerut (M.Sc) 93]

Proof. Let $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ be any two points of H_1. Therefore,

$$\mathbf{c}\mathbf{x}^{(1)} \ge z \text{ and } \mathbf{c}\mathbf{x}^{(2)} \ge z.$$

If $0 \le \lambda \le 1$, then

$$\mathbf{c}\,[\lambda \mathbf{x}^{(1)} + (1-\lambda)\,\mathbf{x}^{(2)}] = \lambda\,(\mathbf{c}\mathbf{x}^{(1)} + (1-\lambda)\,\mathbf{c}\mathbf{x}^{(2)} \ge \lambda z + (1-\lambda)\, z = z$$

Hence $\mathbf{x}^{(1)}, \mathbf{x}^{(2)} \in H_1$ and $0 \le \lambda \le 1$ $\Rightarrow [\lambda \mathbf{x}^{(1)} + (1-\lambda)\,\mathbf{x}^{(2)}] \in H_1$. So H_1 is convex.

Similarly, if $\mathbf{x}^{(1)}, \mathbf{x}^{(2)} \in H_2$, $0 \le \lambda \le 1$, then replacing the inequality sign '$\ge$' by '$\le$' in above, it is true that

$$[\lambda \mathbf{x}^{(1)} + (1-\lambda)\,\mathbf{x}^{(2)}] \in H_2$$

So H_2 is also convex.

Corollary. *The open half spaces* : $\{\mathbf{x} \mid \mathbf{c}\mathbf{x} > z\}$ and $\{\mathbf{x} \mid \mathbf{c}\mathbf{x} < z\}$ *are convex sets.*

Proof. Exercise for the student.

Theorem 4.3. (a) *The intersection of two convex sets is also a convex set.* **[Meerut 90]**

(b) *Intersection of any finite number of convex sets is also a convex set.*

Proof. (a) Let C_1 and C_2 be two convex sets and also let $C = C_1 \cap C_2$.

To show that C is convex.

Let $\mathbf{x}^{(1)}, \mathbf{x}^{(2)} \in C$ and $S = \{\mathbf{x} \mid \mathbf{x} = \lambda \mathbf{x}^{(1)} + (1-\lambda)\,\mathbf{x}^{(2)}, 0 \le \lambda \le 1\}$

Now, $\mathbf{x}^{(1)}, \mathbf{x}^{(2)} \in C \Rightarrow \mathbf{x}^{(1)}, \mathbf{x}^{(2)} \in C_1$ (C_1 being convex)

Also, $\mathbf{x}^{(1)}, \mathbf{x}^{(2)} \in C \Rightarrow \mathbf{x}^{(1)}, \mathbf{x}^{(2)} \in C_2$ (C_2 being convex)

Therefore, $\mathbf{x}^{(1)}, \mathbf{x}^{(2)} \in C \Rightarrow S \subset C_1$ and $S \subset C_2 \Rightarrow S \subset C_1 \cap C_2 \Rightarrow S \subset C$.

Hence C is convex.

(b) Let $C_1, C_2, \ldots, C_n$ be n convex sets and $C = C_1 \cap C_2 \cap \ldots \cap C_n$.

But, $\mathbf{x}_1 \in C_1 \cap C_2 \cap \ldots \cap C_n \Rightarrow \mathbf{x}_1 \in C_i$, for all $i = 1, 2, \ldots, n$

and $\mathbf{x}_2 \in C_1 \cap C_2 \cap \ldots \cap C_n \Rightarrow \mathbf{x}_2 \in C_i$, for all $i = 1, 2, \ldots, n$.

Since C_i is convex set for all $i = 1, 2, \ldots, n$,

$\therefore$ $\mathbf{x}_1, \mathbf{x}_2 \in C_i \Rightarrow \lambda \mathbf{x}_1 + (1-\lambda)\,\mathbf{x}_2 \in C_i$, for all $i = 1, 2, \ldots, n$, where $0 \le \lambda \le 1$

$\Rightarrow \lambda \mathbf{x}_1 + (1-\lambda)\,\mathbf{x}_2 \subseteq C_1 \cap C_2 \cap \ldots \cap C_n$, $0 \le \lambda \le 1$.

That is, $x_1 \in C_1 \cap C_2 \cap \ldots \cap C_n$ and $x_2 \in C_1 \cap C_2 \cap \ldots \cap C_n$

$$\Rightarrow \lambda x_1 + (1-\lambda) x_2 \subseteq C_1 \cap C_2 \cap \ldots\ldots \cap C_n, \ 0 \le \lambda \le 1.$$

Hence by definition $C_1 \cap C_2 \cap \ldots \cap C_n$ is a convex set.

Corollary. *If C_1 and C_2 are closed convex sets, then $C_1 \cap C_2$ is also a closed convex set.*

Q. 1. Give an example of convex set.
2. Write a note on convex sets and their applications to linear programming problems. [Kanpur (B.Sc.) 92]
3. Is it true that the intersection of an arbitrary family of convex sets in R^n is not necessarily convex ?

Illustrative Examples

Example 5. *Show that $S = \{ (x_1, x_2, x_3) : 2x_1 - x_2 + x_3 \le 4, x_1 + 2x_2 - x_3 \le 1 \}$ is a convex set.*

Solution. Obviously, S is the intersection of two half spaces, *viz*,

$$H_1 = \{(x_1, x_2, x_3) : 2x_1 - x_2 + x_3 \le 4\} \text{ and } H_2 = \{(x_1, x_2, x_3) : x_1 + 2x_2 - x_3 \le 1\}$$

Since H_1 and H_2 are convex, so $S = H_1 \cap H_2$ is also convex.

Example 6. *Let A be an $m \times n$ matrix and* **b** *an m-vector, then show that $\{\mathbf{X} \in R^n : \mathbf{AX} \le \mathbf{b}\}$ is a convex set.*

Solution. Let $\mathbf{X} = (x_1, x_2, \ldots, x_n)$, $\mathbf{b} = (b_1, b_2, \ldots, b_n)$ and $\mathbf{A} = (a_{ij})_{m \times n}$, then the set $S = \{\mathbf{X} \in R^n : \mathbf{AX} \le \mathbf{b}\}$ can be represented by m-inequalities:

$$\left.\begin{array}{l} a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n \le b_1 \\ a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n \le b_2 \\ \ldots\ldots \quad \ldots \quad \ldots\ldots \\ a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n \le b_m \end{array}\right\}$$

Thus, the set S is the intersection of m half spaces,

$$H_i = \{(x_1, x_2, \ldots, x_n) : a_{i1} x_1 + a_{i2} x_2 + \ldots + a_{in} x_n \le b_i, i = 1, 2, \ldots, m\}$$

Since each half space is convex, $S = \bigcap_{i=1}^{m} H_i$ is also convex.

4.15-2 Polyhedral Convex Set

Definition. *The intersection of a finite number of closed half spaces in R^n is called a* ***polyhedral convex set.***

Theorem 4.4. *Let S and T be two convex sets in R^n, then $\alpha S + \beta T$ is also convex. $\alpha, \beta \in R$.*

Proof. Let $S \subset R^n$ and $T \subset R^n$ be two convex sets; and $\alpha, \beta \in R$.

Let $\mathbf{X}, \mathbf{Y}$ be two points of $\alpha S + \beta T$.

Then $\mathbf{X} = \alpha \mathbf{U}_1 + \beta \mathbf{V}_1$ and $\mathbf{Y} = \alpha \mathbf{U}_2 + \beta \mathbf{V}_2$ where $\mathbf{U}_1, \mathbf{U}_2 \in S$, and $\mathbf{V}_1, \mathbf{V}_2 \in T$. ...(1)

For any scalar λ, $0 \le \lambda \le 1$, we have

$$\lambda \mathbf{X} + (1-\lambda) \mathbf{Y} = \lambda (\alpha \mathbf{U}_1 + \beta \mathbf{V}_1) + (1-\lambda)(\alpha \mathbf{U}_2 + \beta \mathbf{V}_2)$$

$$= \alpha [\lambda \mathbf{U}_1 + (1-\lambda) \mathbf{U}_2] + \beta [\lambda \mathbf{V}_1 + (1-\lambda) \mathbf{V}_2] \quad \ldots(2)$$

Since S is a convex set, $\mathbf{U}_1, \mathbf{U}_2 \in S \Rightarrow \lambda \mathbf{U}_1 + (1-\lambda)\mathbf{U}_2 \in S, 0 \le \lambda \le 1$, ...(3)

and similarly, $\mathbf{V}_1, \mathbf{V}_2 \in T \Rightarrow \lambda \mathbf{V}_1 + (1-\lambda)\mathbf{V}_2 \in T, 0 \le \lambda \le 1$. ...(4)

Now from (1), (3) and (4) $\lambda \mathbf{X} + (1-\lambda) \mathbf{Y} \in \alpha S + \beta T, 0 \le \lambda \le 1$.

Thus $\mathbf{X}, \mathbf{Y} \in \alpha S + \beta T \Rightarrow [\mathbf{X} : \mathbf{Y}] \subset \alpha S + \beta T$.

Hence $\alpha S + \beta T$ is a convex set.

Corollary. *If S and T be two convex sets in R^n, then $S + T$ and $S - T$ are also convex.*

Q. Given two planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ in R^3, prove that their intersection is a convex set but their union is not.

4·16 EXTREME POINTS OF A CONVEX SET

Definition. *A point* **x** *in a convex set C is called an extreme point if* **x** *cannot be expressed as a convex combination of any two distinct points* $\mathbf{x}^{(1)}$ *and* $\mathbf{x}^{(2)}$ *in C.*

Mathematically, a point $\mathbf{x}$ is an extreme point of a convex set if there *do not* exist other points $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}$ (*i.e.*, $\mathbf{x}^{(1)} \neq \mathbf{x}^{(2)}$) in the set such that

$$\mathbf{x} = \lambda \mathbf{x}^{(2)} + (1 - \lambda)\, \mathbf{x}^{(1)},\ 0 < \lambda < 1. \qquad ...(4\cdot15)$$

We should remember the following properties of extreme point :

(i) An extreme point cannot be 'between' any other two points of the set.

(ii) Obviously, an extreme point is a boundary point of the set.

(iii) Not all boundary points of a convex set are necessarily extreme points.

(iv) Some boundary points may lie between two other boundary points.

For example :

(a) In Fig. 4·22, the polygons which are convex sets have the extreme points as their vertices.

(b) The set $C = \{(x_1, x_2) ;\ x_1^2 + x_2^2 \leq 1\}$ is convex. Every point on the circumference is an extreme point. Thus a convex set may also have infinite number of extreme points.

Q. 1. What do you mean by an extreme point of a convex set ?

2. Give example of convex sets with : (i) Just one extreme point, (ii) Infinitely many extreme points, and (iii) with no extreme points.

3. Define an extreme point of a convex set. Can there be any convex set without any extreme points ? Prove that an extreme point of a convex set is a boundary point of the set.

4·17 CONVEX COMBINATION OF VECTORS

We have already defined the convex combination of any two distinct points $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$. Now we proceed to generalize this definition for any finite number of distinct points.

Definition. *A (linear) convex combination of a finite number of points* $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(m)}$ *is defined as a point*

$$\mathbf{x} = \lambda_1\, \mathbf{x}^{(1)} + \lambda_2\, \mathbf{x}^{(2)} + \ldots + \lambda_m\, \mathbf{x}^{(m)} \qquad ...(4\cdot16)$$

where $\lambda_1, \lambda_2, \ldots, \lambda_m \geq 0$; and $\lambda_1 + \lambda_2 + \ldots + \lambda_m = 1$.

Theorem 4.5. *The set of all convex combinations of a finite number of points* $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_m$ *is a convex set.* **[Agra 98; Meerut 90, 88, 73 (S); IAS (Main) 89]**

Proof. Let
$$C = \left\{\mathbf{x} \mid \mathbf{x} = \sum_{i=1}^{m} \lambda_i\, \mathbf{x}_i,\ \lambda_i \geq 0,\ \sum_{i=1}^{m} \lambda_i = 1\right\}$$

To show that C is a convex set :

Let $\mathbf{x}'$ and $\mathbf{x}''$ be in C, so that

$$\mathbf{x}' = \sum_{i=1}^{m} \lambda'_i\, \mathbf{x}_i \ (\text{where } \lambda'_i \geq 0,\ \sum_{i=1}^{m} \lambda'_i = 1) \text{ and } \mathbf{x}'' = \sum_{i=1}^{m} \lambda_i''\, \mathbf{x}_i \ (\text{where } \lambda_i'' \geq 0,\ \sum_{i=1}^{m} \lambda_i'' = 1)$$

Now consider the vector : $\mathbf{x} = \lambda \mathbf{x}' + (1 - \lambda)\, \mathbf{x}'',\ 0 \leq \lambda \leq 1$

$$= \lambda \sum_{i=1}^{m} \lambda_i' \mathbf{x}_i + (1 - \lambda) \sum_{i=1}^{m} \lambda_i''\, \mathbf{x}_i = \sum_{i=1}^{m} [\lambda \lambda_i' + (1 - \lambda)\, \lambda_i'']\, \mathbf{x}_i = \sum_{i=1}^{m} \mu_i\, \mathbf{x}_i,$$

where $\mu_i = \lambda \lambda_i' + (1 - \lambda)\, \lambda_i''$, $i = 1, 2, \ldots, m$.

Since $0 \leq \lambda \leq 1$, $\lambda_i' \geq 0$, $\lambda_i'' \geq 0$, therefore $\mu_i \geq 0$ for each i. Also,

$$\sum_{i=1}^{m} \mu_i = \sum_{i=1}^{m} [\lambda \lambda_i' + (1 - \lambda)\, \lambda_i''] = \lambda \sum_{i=1}^{m} \lambda_i' + (1 - \lambda) \sum_{i=1}^{m} \lambda_i'' = \lambda + (1 - \lambda) = 1$$

Hence $\mathbf{x}$ will be convex combination of vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$, *i.e.* $\mathbf{x} \in C$.

Thus for each pair of points $\mathbf{x}', \mathbf{x}''$ in C, the line segment joining them is contained in the set C. So C is convex.

Q. 1. Define convex set, extreme points of a convex set, convex combination of vectors.

2. Define a line passing through two points and a line segment joining two points, hyperplane, open and closed half spaces, convex set, extreme point of a convex set and convex combination of a finite number of extreme points.

3. Explain the convex combination of vectors.

4. What is meant by convex combination of vectors ? Prove that the set of all convex combinations of linearly independent vectors is a convex set.
5. Prove that any point on the line segment joining two points in R^n can be expressed as a convex combination of two points. Examine the converse for validity.

4·18 CONVEX-HULL

Definition. *The **convex hull** C(X) of any given set of points X is the set of all convex combinations of sets of points from X.*

In other words, the intersection of all convex sets, containing $X \subset R^n$, is called the ***convex hull*** of X and is danoted by $< X >$.

Symbolically, if $X \subset R^n$, then $< X > \; = \; \cap \; W_i$ where for each $W_i \supset X$ and W_i is a convex set.

Since the intersection of the members of any family of convex sets is convex, it follows that $< X >$, the convex hull of X, is a convex set.

Now for any set $X \subset R^n$, we have :

(i) $< X >$ is a convex set, $X \subset < X >$, and

(ii) if $W \supset X$ be a convex set, then $< X > \subset W$.

Thus the convex-hull of a set $X \subset R^n$ is the smallest convex set containing X.

For example :

(i) If X is just the eight vertices of a cube, then the convex hull C(X) is the whole cube.

(ii) If X is the boundary of a circle, then C(X) is the whole circle.

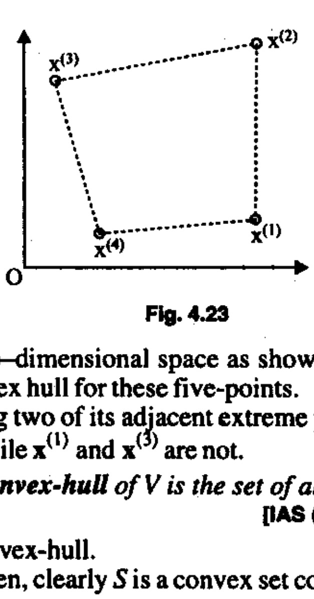


Fig. 4.23

(iii) Let the five points $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \mathbf{x}^{(3)}, \mathbf{x}^{(4)}$ and $\mathbf{x}^{(5)}$ be given in two–dimensional space as shown in Fig. 4·4. Then, the dotted lines represent the boundaries of the convex hull for these five-points.

Furthermore, the ***edge*** of a convex hull is defined as the line joining two of its adjacent extreme points. It should be noted that the two extreme points $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are adjacent while $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(3)}$ are not.

Theorem 4.6. *If V is any finite subset of vectors in R^n, then the **convex-hull** of V is the set of all convex combinations of vectors in V.* **[IAS (Main) 89]**

Proof. Let V be a finite subset of vectors in R^n and $< V >$ be its convex-hull.

Also, let S be the set of all convex combinations of vectors in V. Then, clearly S is a convex set containing V, and hence $< V > \subset S$.

Again $< V >$ contains V. Now show that this implies $S \subset < V >$. To prove this, finite induction on the number, m, of vectors in V will be used.

For $m = 2$, let $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}$ be two vectors in V. Then $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}$ are contained in $< V >$ which is a convex set. Hence $\mathbf{x} = \lambda \mathbf{x}^{(1)} + (1 - \lambda)\, \mathbf{x}^{(2)},\; 0 \le \lambda \le 1$, also lies in $< V >$.

Therefore, all convex combinations of $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}$ lie in $< V >$.

Now assume that, for any positive integer m, the result is true for a set of at most $m - 1$ vectors.

Consider the set $V = \{\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(k)}\}$. Then $< V >$ is a convex set containing V and in particular $\{\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(k-1)}\}$. Let

$$X = \left\{ \mathbf{x} \mid \mathbf{x} = \sum_{i=1}^{m-1} \lambda_i \mathbf{x}^{(i)},\quad \lambda_i \ge 0,\quad \sum_{i=1}^{m-1} \lambda_i = 1 \right\}$$

But, by induction hypothesis $X \subset < V >$.

Also, $< V >$ contains X, as well as $\mathbf{x}^{(m)}$. Therefore, $< V >$ contains all line segments joining $\mathbf{x}^{(m)}$ to $\mathbf{x}^{(1)}$, *i.e.*

$$\mathbf{x} = \mu\, \mathbf{x}^{(m)} + (1 - \mu) \sum_{i=1}^{m-1} \lambda_i \mathbf{x}^{(i)},\quad \lambda_i \ge 0,\quad \sum_{i=1}^{m-1} \lambda_i = 1, 0 \le \mu \le 1$$

is a point in $< V >$, which implies that $\mathbf{x} = \sum_{i=1}^{m} \beta_i \mathbf{x}^{(i)}$, where $\beta_i = (1 - \mu)\, \lambda_i$, $\beta_m = \mu$ for $i = 1, 2, \ldots, m - 1$.

Since $\lambda_i \geq 0$ for each i and $0 \leq \mu < 1$, $\beta_i \geq 0$ for $i = 1, 2, \ldots m$.

Also, $$\sum_{i=1}^{m} \beta_i = \mu + (1-\mu) \sum_{i=1}^{m-1} \lambda_i = 1 .$$

Therefore, $x = \sum_{i=1}^{m} \beta_i x^{(i)}$ is a convex combination of the vectors $x^{(1)}, x^{(2)}, \ldots, x^{(k)}$ and lies in $< V >$. Thus $S \subset < V >$.

Hence $< V > = S$.

Example 7. *Let* $A = \{(X, Y) \subset R^n$, *then show that* $< A > = [X : Y]$.

Solution. Since the line segment $[X : Y]$ is a convex set and $X, Y \in [X : Y]$, so

$$[X : Y] \text{ is a convex set containing } A. \qquad \ldots(1)$$

If $W \subset R^n$ be a convex set containing A, then

$$X, Y \in A \Rightarrow X, Y \in W \Rightarrow [X : Y] \subset W \qquad \ldots(2)$$

From (1) and (2), we have $< A > = [X : Y]$.

Q. 1. Define convex-hull of a set. Prove that the convex-hull of a finite number of points is a convex set. **[Delhi BSc (Maths) 93]**

2. Obtain the convex hull of the boundary of a circle.

4.19 CONVEX POLYHEDRON, CONVEX CONE, SIMPLEX AND CONVEX FUNCTION

Definition. *If the set* **X** *consists of a finite number of points, the convex-hull of* **X** *is called a* ***convex polyhedron.***

Alternatively, *the set of all convex combinations of a finite number of points is called the convex polyhedron spanned by these points.*

For example, convex hull of eight vertices of a cube is a convex polyhedron.

Convex cone. *A non-empty subset* $C \subset R^n$ *is called a* ***cone*** *if for each* $X \in C$, and $\lambda \geq 0$, *the vector* λX *is also in C.*

A cone is called a ***convex cone*** *if it is a convex set.*

For example, if A be an $m \times n$ matrix then the set of n vectors x satisfying the constraint $AX \geq 0$ is a convex cone in R^n. It is a cone because if $A X \geq 0$, then $A(\lambda X) \geq 0$ for $\lambda \geq 0$.

It is convex because if $A x^{(1)} \geq 0$ and $A x^{(2)} \geq 0$, then $A[\lambda x^{(1)} + (1-\lambda) x^{(2)}] \geq 0$.

Simplex. *A simplex is an n-dimensional convex polyhedron having exactly n + 1 vertices.*

For example, a simplex in zero-dimension is a point ; in one-dimension it is a line ; in two-dimension it is a triangle ; and in three-dimension it is a tetrahedron.

Convex Function. *A function* $f(x)$ *is said to be strictly convex at* x *if for any two other distinct points* x_1 and x_2, $f\{\lambda x_1 + (1-\lambda) x_2\} < \lambda f(x_1) + (1-\lambda) f(x_2)$, *where* $0 < \lambda < 1$.

On the other hand, a function $f(x)$ is strictly concave if $-f(x)$ is strictly convex. We are now in a position to prove the result which is very important.

Q. 1. In your words explain the following terms :
(i) Polytope, (ii) Hyperplane, (iii) Simplex, (iv) Convex cone. **[Meerut (MA) 93]**

2. Define : Hyperplane, Simplex, and Convex polydedron.
3. What is meant by convex polyhedron ?
4. For a convex cone, show that the positive sum of any two vectors in the cone is also in the cone.
5. Define a convex polygon. Is every convex set in R^n a polygon also ?
6. Draw a convex polygon and a non-convex polygon.
7. Define a convex function and prove that the sum of two convex functions is also a convex function. **[Virbhadrah 2000]**

4.20 FUNDAMENTAL THEOREM OF LINEAR PROGRAMMING

Theorem 4.7. *The collection of all feasible solutions to L.P. problems constitutes a convex set whose extreme points correspond to the basic feasible solutions.* **[Agra 99; Garhwal 96, 95; Kanpur B.Sc 95, 90]**

Proof. Let F be a set of all feasible solutions of the system $\mathbf{Ax} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}$.

If the set F of solutions has only one element, then F is a convex set. Hence the theorem is true in this case.

Now assume that there are at least two distinct points $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ in F.

Then we have

$$\mathbf{Ax}^{(1)} = \mathbf{b} \quad (\text{for} \quad \mathbf{x}^{(1)} \geq 0) \quad \text{and} \quad \mathbf{Ax}^{(2)} = \mathbf{b} \quad (\text{for} \quad \mathbf{x}^{(2)} \geq 0). \qquad ...(4\cdot15)$$

We only need to show that every convex combination of any two feasible solutions is also a feasible solution.

We define a new point $\mathbf{x}^{(0)}$ as the convex combination of $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$. This implies that

$$\mathbf{x}^{(0)} = \lambda\, \mathbf{x}^{(1)} + (1 - \lambda)\, \mathbf{x}^{(2)}, 0 \leq \lambda \leq 1.$$

By definition, F is convex if $\mathbf{x}^{(0)}$ also belong to F. To show this is true we must show that $\mathbf{x}^{(0)}$ satisfies the system of constraints $\mathbf{Ax} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}$.

Thus, $\mathbf{A}\,\mathbf{x}^{(0)} = \mathbf{A}\,[\,\lambda \mathbf{x}^{(1)} + (1 - \lambda)\, \mathbf{x}^{(2)}\,] = \lambda\, \mathbf{A}\,\mathbf{x}^{(1)} + (1 - \lambda)\, \mathbf{A}\,\mathbf{x}^{(2)} = \lambda \mathbf{b} + (1 - \lambda)\, \mathbf{b} = \mathbf{b}$ [from (4.15)]

Also, since $0 \leq \lambda \leq 1$, $\mathbf{x}^{(1)} \geq 0$, $\mathbf{x}^{(2)} \geq 0$, then $\mathbf{x}^{(0)}$ is also ≥ 0. This means that $\mathbf{x}^{(0)} \in \mathbf{F}$ and consequently F is convex.

Extreme-point correspondence: **[Meerut (L.P.) 90].**

Suppose that $\mathbf{x} = [\mathbf{x_B}, \mathbf{0}]$ is a basic feasible solution, where $\mathbf{x_B}$ is an $m \times 1$ vector, such that for a non-singular submatrix $\mathbf{B}$ of $\mathbf{A}$ we have $\mathbf{Bx_B} = \mathbf{b}$.

If possible let us suppose that $\mathbf{x}$ be a point of F, such that there exist $\mathbf{x}^{(1)}, \mathbf{x}^{(2)} \in F$, so that

$$\mathbf{x} = \lambda \mathbf{x}^{(1)} + (1 - \lambda)\, \mathbf{x}^{(2)}, 0 < \lambda < 1.$$

Let $\mathbf{x}^{(1)} = [\mathbf{u}_1, \mathbf{v}_1]$ and $\mathbf{x}^{(2)} = [\mathbf{u}_2, \mathbf{v}_2]$ where $\mathbf{u}_1, \mathbf{u}_2$ are $m \times 1$ vectors and $\mathbf{v}_1, \mathbf{v}_2$ are $(n - m) \times 1$ vectors. then

$$[\mathbf{x_B}, \mathbf{0}] = \lambda\,[\mathbf{u}_1, \mathbf{v}_1] + (1 - \lambda)\,[\mathbf{u}_2, \mathbf{v}_2]$$

$\therefore \quad \mathbf{x_B} = \lambda \mathbf{u}_1 + (1 - \lambda)\, \mathbf{u}_2$ and $\mathbf{0} = \lambda \mathbf{v}_1 + (1 - \lambda)\, \mathbf{v}_2, 0 < \lambda < 1$.

Since $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}$ are feasible solutions, therefore $\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2 \geq \mathbf{0}$.

Now $\qquad 0 < \lambda < 1$ and $\mathbf{0} = \lambda \mathbf{v}_1 + (1 - \lambda)\, \mathbf{v}_2$.

Therefore we must have $\mathbf{v}_1 = \mathbf{v}_2 = \mathbf{0}$. Thus $\mathbf{x}^{(1)} = [\mathbf{u}_1, \mathbf{0}]$, $\mathbf{x}^{(2)} = [\mathbf{u}_2, \mathbf{0}]$.

Again, since $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}$ satisfy $\mathbf{Ax} = \mathbf{b}$, we have $\mathbf{Bu}_1 = \mathbf{b}$ and $\mathbf{Bu}_2 = \mathbf{b}$.

Also since $B\mathbf{x_B} = \mathbf{b}$ since expression of $\mathbf{b}$ as linear combination of basis vectors must be unique, therefore

$$\mathbf{u}_1 = \mathbf{u}_2 = \mathbf{x_B}.$$

Hence $\mathbf{x} = \mathbf{x}^{(1)} = \mathbf{x}^{(2)}$. This is a contradiction for $\mathbf{x}^{(1)} \neq \mathbf{x}^{(2)}$. Hence $\mathbf{u}$ is an extreme point of F.

Thus the theorem is completely proved.

Remark. The above theorem indicates that there is only one extreme point for a given feasible solution, and one basic feasible solution, corresponds to any one extreme point. Thus, it can be concluded that the number of extreme points of the feasible region is finite and that it cannot exceed the number of its basic solutions

(of course, in absence of degeneracy, there is a one to-one correspondence between the extreme points and the basic feasible solutions). Hence the maximum number of extreme points is given by

$$^{m+n}C_m = \frac{(m+n)\,!}{n\,!\,m\,!}.$$

Q. 1. Prove that the collection of all feasible soluitons of a linear programming problem constitutes a convex set, whose extreme points correspond to basic feasible solutions. **[Meerut (Stat.) 95]**

2. Prove that basic feasible solutions of a linear programming problem correspond to the extreme points of the convex set generated by the set of feasible solutions of the linear programming problem. **[IAS (Maths.) 98; Meerut (B.Sc.) 90]**

3. Prove that every extreme point of the convex set of all feasible solutions of the system $Ax = \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$ is a basic feasible solution.

Theorem 4.8. *If the convex set of the feasible solutions of* $\mathbf{AX} = \mathbf{b}, \mathbf{b} \geq \mathbf{0}$, *is a convex-polyhedron, then at least one of the extreme points gives an optimal solution.*

If the optimal solution occurs at more than one extreme point, the value of the objective function will be the same for all convex combinations of these extreme points.

Proof. ***First Part.*** Let $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(k)}$ be the extreme points of the feasible region F of the LP problem :

$$\text{Max. } z = \mathbf{cx}, \text{ subject to } \mathbf{Ax} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}.$$

Suppose $\mathbf{x}^{(m)}$ is the extreme point among $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(k)}$ at which the value of the objective function is maximum (say, z^*).

Then,
$$z^* = \max_{t=1,2,\ldots,k} \{\mathbf{cx}^{(t)}\} = \mathbf{cx}^{(m)}. \qquad \ldots(4.16)$$

We now consider a point, $\mathbf{x}^{(0)}$ in F which is not an extreme point and let $z^{(0)}$ be the corresponding value of the objective function. Then,

$$z^{(0)} = \mathbf{cx}^{(0)}. \qquad \ldots(4.17)$$

Since $\mathbf{x}^{(0)}$ is not an extreme point, it can be expressed as a convex combination of the extreme points $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(k)}$ of the feasible region F, where F is assumed to be a bounded set. Then,

$$\mathbf{x}^{(0)} = \lambda_1 \mathbf{x}^{(1)} + \lambda_2 \mathbf{x}^{(2)} + \ldots + \lambda_k \mathbf{x}^{(k)}, \quad \text{where} \quad \lambda_1, \lambda_2, \ldots, \lambda_k \geq 0, \ \sum_{t=1}^{k} \lambda_t = 1.$$

Thus, substituting the value of $\mathbf{x}^{(0)}$ in (4·3), we get

$$z^{(0)} = \mathbf{c}\,(\lambda_1 \mathbf{x}^{(1)} + \lambda_2 \mathbf{x}^{(2)} + \ldots\ldots + \lambda_k \mathbf{x}^{(k)}) \leq \mathbf{c}\,\mathbf{x}^{(m)}$$

or
$$z^{(0)} \leq z^* \ [\text{since } \mathbf{cx}^{(m)} = z^* \text{ from (4·2)}]$$

which implies that, at optimum solution, the extreme point solution is at least as good as any other feasible solution in F.

Second Part.

Let $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(r)}$ $(r \leq k)$ be the extreme points of the feasible region F at which the objective function assumes the same optimum value. This means, $z^* = \mathbf{cx}^{(1)} = \mathbf{cx}^{(3)} = \ldots = \mathbf{cx}^{(r)}$.

Further, let $\mathbf{x} = \lambda_1 \mathbf{x}^{(1)} + \lambda_2 \mathbf{x}^{(2)} + \ldots \lambda_r \mathbf{x}^{(r)}$; $\lambda_1, \lambda_2, \ldots, \lambda_r \geq 0$; $\sum_{j=1}^{r} \lambda_j = 1$ be the convex combination of $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(r)}$, then

$$\mathbf{cx} = \mathbf{c}\,[\lambda_1 \mathbf{x}^{(1)} + \lambda_2 \mathbf{x}^{(2)} + \ldots + \lambda_r \mathbf{x}^{(r)}] = \lambda_1 (\mathbf{cx}^{(1)}) + \lambda_2 (\mathbf{cx}^{(2)}) + \ldots + \lambda_r (\mathbf{cx}^{(r)})$$
$$= \lambda_1 z^* + \lambda_2 z^* + \ldots + \lambda_r z^* \qquad = (\lambda_1 + \lambda_2 + \lambda_3 + \ldots + \lambda_r)\, z^*$$

or
$$\mathbf{cx} = z^* \ (\text{since} \sum_{j=1}^{r} \lambda_j = 1).$$

Hence the theorem is proved.

Analytic Method

4.21 ANALYTIC (TRIAL AND ERROR) METHOD

Graphical method is not applicable to the LP problems having more than two variables. In such cases, the *analytic solution* (which is also called the *trial and error method*), may seem to be useful. The concepts of analytic solution are also important to gain a good grasp over the more powerful ***simplex technique*** (discussed in the next chapter).

Example 1. Consider the LP problem of ***Example 29*** (Page 77) :

$$\text{Max. } z = -0{\cdot}10x_1 + 0{\cdot}50x_2 \text{ subject to, } 2x_1 + 5x_2 \leq 80 \qquad x_1 + x_2 \leq 20 \text{ and } x_1, x_2 \geq 0.$$

Introducing the slack variables x_3 and x_4, the problem becomes

$$\text{Maximize } z = 0{\cdot}50x_2 - 0{\cdot}10x_1 + 0x_3 + 0x_4, \text{ subject to}$$

$$\left.\begin{aligned} 2x_1 + 5x_2 + x_3 &= 80 \\ x_1 + x_2 + x_4 &= 20 \end{aligned}\right\}, \text{ and } x_1, x_2, x_3, x_4 \geq 0. \qquad \ldots(4.18)$$

For a system of m equations in n variables (when $n > m$) a solution in which at least $(n - m)$ of the variables have the value zero is a *vertex*. This solution is called a ***basic solution.***

To determine the basic solutions of the system (4·4) put $n - m$ variables equal to zero at a time and solve the resulting system of equations to obtain a basic solution. Here $n = 4$ (the number of variables) and $m = 2$) the number of equations). So $n - m = 2$ variables should be zero at a time. This can be done in ${}^4C_2 = 6$ number of ways. Hence at the most there will be 6 basic solutions which can be obtained as follows :

Set 1. *When* $x_1 = x_2 = 0$, system (4·18) gives the basic solution $x_3 = 80$, $x_4 = 20$.

Set 2. *When* $x_3 = x_4 = 0$, system (4·18) becomes : $2x_1 + 5x_2 = 80$ and $x_1 + x_2 = 20$ which on solving gives the basic solution $x_1 = 20/3$, $x_2 = 40/3$.

Set 3. *When* $x_2 = x_3 = 0$, the system (4·18) gives the basic solution : $x_1 = 40$, $x_4 = -20$, which is *infeasible* also.

Set 4. *When* $x_2 = x_4 = 0$, the system (4·18) gives the basic solution : $x_1 = 20$, $x_3 = 40$.

Set 5. *When* $x_1 = x_4 = 0$, the system (4·18) gives the basic solution : $x_2 = 20$, $x_3 = -20$, which is also *infeasible*.

Set 6. *When* $x_1 = x_3 = 0$, the system (4·17) gives the basic solution : $x_2 = 16$, $x_4 = 4$.

Substituting the values of basic variables in the objective function, the corresponding values of z are obtained as below :

Set	*Basic Solution* (x_1, x_2, x_3, x_4)	*Objective Function* $z = -0{\cdot}10x_1 + 0{\cdot}50x_2 + 0x_3 + 0x_4$
(1)	(0, 0, 80, 20)	0
(2)	(20/3, 40/3, 0, 0)	6
(3)*	(40, 0, 0, – 20)	Infeasible
(4)	(20, 0, 40, 0)	– 2
(5)*	(0, 20, – 20, 0)	Infeasible
(6)	(0, 16, 0, 4)	8

Since solution-sets (3)* and (5)* yield a negative coordinate, each contradicting thereby the non-negativity constraints, these are infeasible and so are dropped from the consideration. The optimum solution thus obtained is : $x_1 = 0$, $x_2 = 16$, max $z = 8$ as obtained earlier also by graphical method.

Further, it is important to observe that four basic feasible solution sets (1), (2), (4) and (6) exactly coincide with the corner points O, E, C and B of the feasible region (see Fig. 4.7) respectively, and one of these corner points gives the optimal solution.

Extreme point theorem also states that *an optimal solution to an LP problem occurs at one of the vertices of the feasible region.*

Since the vertices of the feasible region are corresponding to the *basic feasible solutions*, the objective function is optimal at least at one of the basic solutions. Some of the vertices may be infeasible which are dropped from consideration.

Disadvantages of analytical method :

1. In LP problems in which m and n are large, solution of various sets of simultaneous equations become extremely difficult and time consuming.
2. Some of the sets give infeasible solutions also. There should be some technique to detect all such sets and not solve them at all.
3. As seen from above table, the value of z jumps from 0 to 6 to – 2 to 8, *i.e.*, there are up's and down's.

These disadvantages are overcome by *simplex method* yielding successive solutions with progressively improving the value of z, culminating into the optimal one.

Q. 1. Explain the procedure of generating extreme point solutions to a linear programming problem pointing out the assumptions made, if any.

2. Compute all the basic feasible solutions of the LP problem : Max $z = 2x_1 + 3x_2 + 4x_3 - 7x_4$ s.t. $2x_1 + 3x_2 - x_3 + 4x_4 = 8$
$x_1 - 2x_2 + 6x_3 - 7x_4 = -3$
and choose that one which maximizes z.

[IAS (Main) 2001 (Type); Rewa (MP) 93 ; Roorkee (B.E. IVth) 91 ; Meerut (B.Sc.) 90]

SELF-EXAMINATION PROBLEMS

1. Prove that a vertex is a boundary point but all boundary points are not vertices. Give examples. Identify the vertices, if any, of the following sets :
   (a) $\{x : |x| \le 1, x \in R^n\}$ (b) $\{x : x = (1-\lambda) x_1 + \lambda x_2, \lambda \ge 0, x_1, x_2 \in R^n\}$.
   [**Ans.** (a) Vertex is 1, (b) The point x will be vertex when there does not exist any pair of points x_1, x_2 which satisfies the given conditions for $0 < \lambda < 1$]
2. Which of the following sets are convex; if so, why ?
   (i) $X = \{(x_1, x_2) ; x_1x_2 \le 1, x_1 \ge 0, x_2 \ge 0\}$
   (ii) $X = \{(x_1, x_2) : x_2^2 - 3 \ge - x_1^2 : x_1 \ge 0$ and $x_2 \ge 0\}$ **[Delhi BSc. 93]**
   (iii) $X = \{(x_1, x_2) : x_1^2 + x_2^2 \le 9\}$,
   (iv) $X = \{(x_1, x_2) : x_2^2 \le 4x_1 ; x_1, x_2 \ge 0\}$ **[Delhi B.Sc. 90]**
   [**Ans.** (i) Concave (ii) Not Convex (iii) Convex (iv) Convex]
3. If a linear programming problem, Max **AX** = **b** , **X** ≥ **0**, has at least an optimal feasible solution, then at least one basic feasible solution must be optimal. **[Meerut (B.Sc.) 90]**
4. When is a set K ⊂ E^n said to be convex ? Show that for a set K to be convex it is necessary and sufficient that every convex linear combination of points in K belong to K. What is the role of the theory of convex sets in the solution of linear programming problems ? **[I.A.S. (Maths) 90]**
5. Define a convex function f on a convex set S in R^n. Show that the points of S on which f takes on its global minimum is a convex set. **[Agra M.Sc. (Math) 98]**
6. Compute all basic fecible solutions of the linear programming problem :
   Max. $z = 2x_1 + 3x_2 + 2x_3$, subject to $2x_1 + 3x_2 - x_3 = 8$, $x_1 - 2x_2 + 6x_3 = -3$, $x_1, x_2, x_3 \ge 0$ and hence indicate the optimum solution. **[IAS (Main) 2001]**

MODEL OBJECTIVE QUESTIONS

FILL-UP THE BLANKS

1. Fill in the blanks so that the following statements are correct. Write only the answers.
   (i) Any solution to an LPP which satisfies the non-negativity condition is called a ......
   (ii) A feasible solution to an LPP which optimizes the objective function is called ....
   (iii) A constraint which does not affect the solution to an LPP is called a ....
   (iv) The variables corresponding to the columns of a basis matrix are known as ....
   (v) In an LPP having m constraints in n variables, $m < n$, the maximum number of B.F. solutions is ....
   (vi) In usual notations, a BFS to an LPP
   max $z = cx$ subject to $Ax = b$, $x \ge 0$,
   is given by x_B = ...
   (viii) A BFS to an LPP is said to be optimal if it .... the objective fûnction.
2. Each of the following statements is either **true** or **false**. Indicate your choice of the answer by writing TRUE or FALSE for each statement.
   (i) For an LPP with m constraints in n variables it is necessary that $m < n$.
   (ii) An LPP in three variables cannot be solved graphically.
   (iii) The feasible region to the LPP : max $z = 3x_1 - 2x_2$ subject to $x_1 + x_2 \le 1$, $2x_1 + 2x_2 \ge 4$ and $x_1, x_2 \ge 0$ is empty.
   (iv) If an LPP has an unbounded feasible solution, then it may have bounded optimal solution.
   (v) In the standard form of an LPP all the constraints are in the form of equations.
   (vi) The number of basic feasible solutions cannot exceed the number of basic solutions.
3. For each of the following statements, one of the four alternatives is correct. Indicate your choice of correct answer for each statement by writing one of the letters **a**, **b**, **c**, **d** whichever is appropriate.
   (i) The feasible region to an LPP in two variables depends upon
   (a) objective function. (b) constraints and non-negativity conditions.
   (c) constrainst only. (d) non-negativity conditions only.
   (ii) The feasible region represented by the constraints
   $x_1 + x_2 \le 1$, $3x_1 + x_2 \ge 3$, $x_1 \ge 0$, $x_2 \ge 0$, of the objective function $z = x_1 + 2x_2$ is :
   (a) a polygon. (b) unbounded set.
   (c) empty set. (d) a singleton set.

(iii) The feasible region represented by the constraints

$x_1 + x_2 \le 1$, $-3x_1 + x_2 \ge 3$ and non-negativity conditions $x_1, x_2 \ge 0$ is :

(a) a polygon. (b) unbounded set. (c) empty set. (d) a singleton set.

(iv) The solution $x_1 = 1$, $x_2 = \frac{1}{2}$, $x_3 = x_4 = x_5 = 0$ to the equations :

$x_1 + 2x_2 + x_3 + x_4 = 2$ and $x_1 + 2x_2 + \frac{1}{2}x_3 + x_5 = 2$ is

(a) a basic solution. (b) a non-basic solution.
(c) a non-degenerate FS. (d) none of these.

Answers

1. (i) Feasible solution (ii) An optimal solution
(iii) Redundant (iv) Basic variables
(v) ^nC_m, *i.e.*, $n!/[m!(n-m)!]$ (vi) B^{-1}b
(vii) Optimizes.
2. (i) False (ii) False (iii) True (iv) True (v) True (vi) True
3. (i) b (iii) d (iv) c (iv) b.

TRUE OR FALSE QUESTIONS

State whether the following statement are 'True' or 'False',

1. Linear programming deals with problems involving only a single objective. (T F)
2. LP problem can be tackled only in those situations where the constraints and the objective function can be stated in terms of linear expessions. (T F)
3. Linear programming takes into consideration the effect of time and uncertainty. (T F)
4. The proportionality property in LP is not satisfied when the per unit contribution of a variable in the objective function is dependent on the value of the variables. (T F)
5. The requirement of a linear program which makes it "linear" is that the objective function and the constraints be expressible as linear equalities or inequalities. (T F)
6. The graphic method of solving linear programmes is useful because of its applicability to problems with many variables. (T F)
7. Total contribution is influenced by volume of sales but not by fixed costs. (T F)
8. Any solution which satisfies at least one of the constraints in a linear programme is included in the feasible region. (T F)
9. The inter-section of any two constraints is an extreme point which is a corner of the reasible region. (T F)
10. The non-negativity conditions mean that all decision variables must be positive. (T F)
11. Since fractional values for decision variables may not be physically meaningful, in practice (for the purpose of implementation) we often round the optimal LP solution to interger values. (T F)
12. All the constraints in an LP are inequalities. (T F)
13. Properly defining the decision variables is an important step in model formulation. (T F)
14. The way in which a problem has been formulated as a model is of considerable interest to the manager, who may one day have to pass judgement on the validity of the model. (T F)
15. An LP problem never has more than one optimal solution. (T F)
16. An LP problem does not possess any feasible solution, if it has no solution that satisfies all constraints. (T F)
17. Constraints appear as straight lines when plotted on a graph. (T F)
18. An LP problem is said to have an unbounded solution, if its solution is not permitted to be infinitely large. (T F)
19. The feasible region is the set of all points that satisfy at least one constraint. (T F)
20. In two-dimensional problems, the intersection of any two constraints gives an extreme point of the feasible region. (T F)
21. An optimal solution uses up all of the limited resources availables. (T F)
22. A well-formulated model will be neither unbounded nor infeasible. (T F)
23. Infeasibility, as opposed to unboundedness, has nothing to do with the objective function. (T F)
24. If an LP is not infeasible, it will have an optimal solution. (T F)
25. Consider any point on the boundary of the feasible region. Such a point satisfies all the contraints. (T F)

Answers

1. T	2. T	3. F	4. T	5. T	6. F	7. T	8. F	9. F
10. F	11. F	12. T	13. T	14. T	15. T	16. T	17. T	18. F
19. F	20. F	21. F	22. T	23. T	24. F	25. T.		

MULTIPLE CHOICE QUESTIONS – I

Examine which of the following alternative is correct in multiple choice questions.

1. Constraints may represent
   (a) limitations. (b) requirements.
   (c) balance conditions. (d) all of the above.
2. A constraint limit the values that
   (a) the objective function can asume. (b) the decision variables can assume.
   (c) neither of the above. (d) both (a) and (b).
3. Linear programming is
   (a) a constrained optimization model. (b) a constrained decision-making model.
   (c) a mathematical programming model. (d) all of the above.
4. Model formulation is important because
   (a) it enables us to use algebraic techniques.
   (b) in a business context, most managers prefer to work with formal models.
   (c) it forces management to address a clearly defined problem.
   (d) it allows the manager to better communicate with the management scientist and therefore to be more discriminating in hiring policies.
5. The non-negativity requirement is included in an LP because :
   (a) it makes the model easier to solve.
   (b) it makes the model correspond more closely to the real-world problem.
   (c) both (a) and (b).
   (d) neither of the above.
6. The distinguishing features of an LP (as opposed to more general mathematical programming models) is :
   (a) the problem has an objective function and constraints.
   (b) all functions in the problem are linear.
   (c) optimal values for the decision variables are produced.
   (d) all of the above.
7. In an LP maximization model
   (a) the objective function is maximized.
   (b) the objective function is maximized and then it is determined whether or not this occurs at an allowable decision.
   (c) the objective function is maximized over the allowable set of decisions.
   (d) all of the above.
8. All variables in the solution of a linear programming problem are either positive or zero because of the exitence of
   (a) An objective function. (b) Strucutral constraints.
   (c) Limited resources. (d) None of the above.
9. Which of the following is not a major requirement of a linear programming problem ?
   (a) There must be alternative courses of action among which to decide.
   (b) An objective for the firm must exist.
   (c) The problem must be of the maximization type.
   (d) Resources must be limited.
10. An iso-profit line represents
    (a) an infinite number of solutions all of which yield the same profit.
    (b) an infinite number of solutions all of which yield the same costs.
    (c) none of (a) and (b).
11. Every corner of the feasible region is defined by
    (a) the intersection of 2 constraint lines.
    (b) some subject of constraint lines and non-negativity conditions.
    (c) neither of the above.
12. The graphical method is useful because
    (a) it provides a general way to solve LP problems.
    (b) it gives geometric insight into the model and the meaning of optimality.
    (c) both (a) and (b).

13. An unbounded feasible region
   (a) arises from an incorrect formulation.
   (b) means the objective function is unbounded.
   (c) neither of the above.
   (d) (a) and (b).
14. Consider an optimal solution to an LP. Which of the following must be true ?
   (a) At least one constraint (not including non-negativity conditions) is active at the point.
   (b) Exactly one constraint (not including non-negativity conditions) is active at the point.
   (c) Neither of the above.
   (d) All of the above.
15. The phrase "unbounded LP" means that
   (a) at least one decision variables can be made arbitrary large without leaving the feasible region.
   (b) the objective contours can be moved as far as desired, in the optimizing direction, and still touch at least one point in the constraint set.
16. Which of the following assertions is true of an optimal solution to an LP ?
   (a) Every LP has an optimal solution.
   (b) The optimal solution always occurs at an extreme point.
   (c) The optimal solution uses up all resources.
   (d) If an optimal solution exists, there will always be at least one at a corner.
   (e) All of the above.
17. An iso-profit contour represents
   (a) an infinite number of feasible points, all of which yield the same profit.
   (b) an infinite number of optimal solutions.
   (c) an infinite number of decisions, all of which yield the same profit.
   (d) none of the above.

Answers

1. (d)	2. (d)	3. (d)	4. (d)	5. (b)	6. (b)	7. (c)	8. (d)	9. (c)
10. (a)	11. (b)	12. (b)	13. (a)	14. (c)	15. (b)	16. (d)	17. (c).	

MULTIPLE CHOICE QUESTIONS – II

1. Mathematical model of LP problem is important because
   (a) it helps in converting the verbal description and numerical data into mathematical expression.
   (b) decision-makers prefer to work with formal models.
   (c) it captures the relevant relationship among decision factors.
   (d) it enables the use of algebraic technique.
2. Linear programming is a
   (a) constrained optimization technique. (b) technique for economic allocation of limited resources.
   (c) mathematical technique. (d) all of the above.
3. A constraint in an LP model restricts
   (a) value of objective function. (b) value of a decision variable.
   (c) use of the available resource. (d) all of the above.
4. The distinguishing feature of an LP model is
   (a) relationship among all variables is linear. (b) it has single objective function and constraints.
   (c) value of decision variables is non-negative. (d) all of the above.
5. Constraints in an LP model represents
   (a) limitations. (b) requirements.
   (c) balancing limitations and requirements. (d) all of the above.
6. Non-negativity condition is an important component of LP model because
   (a) variables value should remain under the control of decision-maker.
   (b) value of variables make sense and correspond to real-world problems.
   (c) variables are interrelated in terms of limited resources.
   (d) none of the above.
7. Before formulating a formal LP model, it is better to
   (a) express each constraint in words. (b) express the objective function in words.
   (c) decision variables are identified verbally. (d) all of the above.
8. Each constraint in an LP model is expressed as an
   (a) inequality with ≥ sign. (b) inequality with ≤ sign.
   (c) equation with = sign. (d) none of the above.

9. Maximization of objective function in LP model means
(a) value occurs at allowable set of decisions.
(b) highest value is chosen among allowable decisions.
(c) neither of above
(d) both (a) and (b).

10. Which of the following is not a characteristic of LP model.
(a) alternative courses of action.
(b) an objective function of maximization type.
(c) limited amount of resources.
(d) non-negativity condition on the value of decision variables.

11. The graphical method of LP problem uses
(a) objective function equation.
(b) constraint equations.
(c) linear equations.
(d) all of the above.

12. A feasible solution to an LP problem
(a) must satisfy all of the problem's constraints simultaneously.
(b) need not satisfy all of the constraints, only some of them.
(c) must be a corner point of the feasible region.
(d) must optimize the value of the objective function.

13. Which of the following statements is true with respect to the optimal solution of an LP problem
(a) every LP problem has an optimal solution.
(b) optimal solution of an LP problem always occurs at an extreme point.
(c) at optimal solution all resources are used completely.
(d) if an optimal solution exists, there will always be at least one at a corner.

14. An iso-profit line represents
(a) an infinite number of solutions all of which yield the same profit.
(b) an infinite number of solutions all of which yield the same cost.
(c) an infinite number of optimal solutions.
(d) a boundary of the feasible region.

15. If an iso-profit line yielding the optimal solution coincides with a constraint line, then
(a) the solution is unbounded.
(b) the solution is infeasible.
(c) the constraint which coincides is redundant.
(d) none of the above.

16. While plotting constraints on a graph paper, terminal points on both the axes are connected by a straight line because
(a) the resources are limited in supply.
(b) the objective function is a linear function.
(c) the constraints are linear equations or inequalities.
(d) all of the above.

17. A constraint in an LP model becomes redundant because
(a) two iso-profit lines may be parallel to each other.
(b) the solution is unbounded.
(c) this constraint is not satisfied by the solution value.
(d) none of the above.

18. If two constraints do not intersect in the positive quadrent of the graph, then
(a) the problem is infeasible.
(b) the rolution is unbounded.
(c) one of the constraints is redundant.
(d) none of the above.

19. Constraints in LP problem are called active if they
(a) represent optimal solution.
(b) at optimality do not consume all the available resources.
(c) both of (a) and (b).
(d) none of the above.

20. The solution space (region) of an LP problem is unbounded due to :
(a) an incorrect formulation of the LP model.
(b) objective function is unbalanced.
(c) neither (a) nor (b).
(d) both (a) and (b).

21. Solution to $z = 4x_1 + 6x_2$, subject to $x_1 + x_2 \le 4$, $3x_1 + x_2 \le 12$, $x_1, x_2 \ge 0$, is
(a) unique. (b) unbounded. (c) degenerate. (d) infinite.
[IES (Mech.) 1992]

22. Which of the following conditions are necessary for applying linear programming ?
1. There must be a well defined objective function.
2. The decision variabls should be interrelated and non-negative.
3. The resources must be in limited supply.
(a) 1 and 2 only. (b) 1 and 3 only. (c) 2 and 3 only. (d) 1, 2 and 3.
[IES (Mech.) 1992]

23. Consider the following statements :
Linear programming model can be applied to
1. line balancing problem. 2. transportation probelm. 3. project management.

Of these statements
(a) 1, 2 and 3 are correct. (b) 1 and 2 are correct. (c) 2 and 3 are correct. (d) 1 and 3 are correct.
[IES (Mech.) 1993]

24. A feasible solution to the linear programming problem should
(a) satisfy the problem constraints.
(b) optimise the objective function.
(c) satisfy the problem constraints and non-negativity restrictions.
(d) satisfy the non-negativity restructions. **[IES (Mech.) 1994]**

25. A variable which has no physical meaning, but is ased to obtain an initial basic feasible solution to the linear programming problem is referred to as
(a) Basic variable. (b) Non-basic variable. (c) Artificial variable. (d) Basis.
[IES (Mech.) 1998]

26. Consider the following linear programming probelm :
Max. $z = 2A + 3B$, subject to $A + B \leq 10$, $4A + 6B \leq 30$, $2A + B \leq 17$, $A, B \geq 0$.
What can one say about the solution ?
(a) It may contain alternative optima. (b) The solution will be unbounded.
(c) The solution will be degenerate. (d) It must be solved by simplex method.
[IES (Mech.) 1997]

27. Consider the following statements regarding the characteristics of the standard form of a linear programming :
1. All the constraints are expressed in the form of equations.
2. The right hand side of each constaint equation is non-negative.
3. All the decision variables are non-negative.

Which of the following statements are correct ?
(a) 1, 2 and 3. (b) 1 and 2. (c) 2 and 3. (d) 1 and 3.
[IES (Mech.) 1999]

28. In case of solution of a two variable linear programming problem by graphical method one constraint line comes parallel to the objective function line. Which one of the following is correct ?
The problem will have :
(a) infeasible solution (b) unbounded soloution
(c) degenerate solution (d) infinite number of optimal solutions. **[IES 2004]**

Answers

1. (a)	**2.** (d)	**3.** (d)	**4.** (a)	**5.** (d)	**6.** (b)	**7.** (d)	**8.** (d)	**9.** (a)
10. (b)	**11.** (b)	**12.** (b)	**13.** (b)	**14.** (b)	**15.** (c)	**16.** (a)	**17.** (b)	**18.** (c)
19. (b)	**20.** (c)	**21.** (a)	**22.** (d)	**23.** (d)	**24.** (c)	**25.** (b)	**26.** (a)	**27.** (a)
28. (b).								

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